MY RESEARCH

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Below I will try to explain (at least elements of) my research. Most of my research is, in some way, related to module theory. A (left) module over a (unital) ring R is an abelian group M equipped with a multiplication $R \times M \to M$ satisfying the axioms of a vector space over a (skew) field. As linear algebra is concerned with the theory of modules over a field (= vector spaces), the study of modules over more general rings can be viewed simply as a sort of generalized linear algebra.

Why study modules? One answer to this question is that the notion of modules captures many important algebraic concepts. The following (far from complete) list of equivalences of categories illustrates the case in point:

$$\begin{array}{ccc}
\left\{\text{Abelian groups}\right\} & \longleftrightarrow & \left\{\text{Modules over the ring } R = \mathbb{Z}\right\} \\
\left\{\begin{array}{c}\text{Linear representations of}\\ \text{a group } G \text{ (over a field } k)\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\text{Modules over the group}\\ \text{algebra } R = kG\end{array}\right\} \\
\left\{\begin{array}{c}\text{Representations of a Lie}\\ \text{algebra } \mathfrak{g} \text{ (over a field } k)\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\text{Modules over the universal}\\ \text{enveloping algebra } R = U(\mathfrak{g})\end{array}\right\} \\
\left\{\begin{array}{c}\text{Smooth vector bundles over a}\\ \text{connected smooth manifold } M\end{array}\right\} & \xleftrightarrow{\text{Swan's theorem}} & \left\{\begin{array}{c}\text{Finitely generated projective}\\ \text{modules over the ring } R = C^{\infty}(M)\end{array}\right\}.
\end{array}$$

What is the goal? Just as every group theorist's dream is (I imagine) to classify all finite groups, the goal in module/representation theory is to classify all modules, up to isomorphism, over any given ring. This is of course an impossible task, so often people restrict their attention to various special kinds of rings/modules and try to study/understand these. To make this more concrete, let us consider the case of:

Cohen–Macaulay rings and modules. One family of rings that appear in algebraic geometry, combinatorics, invariant theory, and commutative algebra—and therefore has been studied intensively—is the class of Cohen–Macaulay (CM) rings. E.g. $R = k[\![x,y]\!]/(x^2)$ is such a ring. For a local CM ring R, the category $\operatorname{mod} R$ of *all* finitely generated R-modules can be built, in some sense, out of the subcategory $\operatorname{MCM} R \subset \operatorname{mod} R$ of maximal CM modules (these are the R-modules M for which a certain numerical invariant, $\operatorname{depth}_R M$, attain the maximal possible value). Hence much attention has been given to this subcategory.

In some cases, it is possible to list all indecomposable objects in $\operatorname{MCM} R$. For example, for the ring $R = k[\![x,y]\!]/(x^2)$ a complete list of non-isomorphic indecomposable maximal CM modules consists of the ideals $I_0 = R$, $I_n = (x, y^n)$ for $n > 0$, and $I_\infty = (x)$ [Buchweitz–Greuel–Schreyer, 1987].

Personally I have been more interested in computing various kinds of invariants of the category $\operatorname{MCM} R$. For example, since it is an exact category, it makes sense to ask for its K-groups $\mathrm{K}_n(\operatorname{MCM} R)$ in the sense of [Quillen, 1972]. When R has finite MCM-representation type, [Yoshino, 1990] computed the Grothendieck group $\mathrm{K}_0(\operatorname{MCM} R)$. In [Holm, 2015] the work of Yoshino is continued, and it is shown that

$$\mathrm{K}_1(\mathrm{MCM}\,R) \cong \mathrm{Aut}_R(M)_{\mathrm{ab}}/\Xi\,,$$

where M is any representation generator of MCM R and Ξ is a subgroup determined by the Auslander–Reiten sequences. For $R = k[\![t^2,t^3]\!]$ one has $M = R \oplus (t^2,t^3)$ and the formula above yields $\mathrm{K}_1(\mathrm{MCM}\,R) \cong k[\![t]\!]^\times$ (the group of units in the power series ring $k[\![t]\!]$).

Another invariant I have computed in [Holm, 2016] is the global dimension of the category MCM R (by which I mean the global dimension of the abelian category of finitely presented additive functors $\mathrm{MCM}\,R \to \mathrm{Ab}$). There are inequalities:

$$d \leqslant \mathrm{gldim}(\mathrm{MCM}\,R) \leqslant \max\{2,d\} \quad \text{where} \quad d = \dim R\,.$$

In particular, $\mathrm{gldim}(\mathrm{MCM}\,R) = d$ if $d \geqslant 2$. Furthermore, $\mathrm{gldim}(\mathrm{MCM}\,R) = 0$ iff R is a field, and $\mathrm{gldim}(\mathrm{MCM}\,R) = 1$ iff R is a discrete valuation ring.

In another paper [Holm, 2017] I give a description of the R-modules that can be written as a direct limit of modules from MCM R.

Homological algebra and homological conjectures. I study modules (and the like) by using methods from homological algebra. This entails that I often need to understand the meaning of (or what kind of information that lies hidden in) certain homological conditions. The long-standing Auslander–Reiten Conjecture involves one such homological condition (in this case: vanishing of Ext). It asserts that every finitely generated module M over an artin algebra R that satisfies $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ for all $i > 0$ must be projective. I have been interested in the connection between various homological conjectures, and in e.g. [Christensen–Holm, 2010] we prove the following implications for any ring R:

R satisfies the Gorenstein Symmetry Conjecture	$\Longleftarrow$	R satisfies the Auslander Conjecture	$\Longrightarrow$	R satisfies the Auslander–Reiten Conjecture .

(The conjectures to the left and right are open. There are counterexamples to the conjecture in the middle [Jorgensen–Şega, 2003], however, many rings do satisfy this conjecture.)

Cotorsion pairs. Consider in a Euclidian space $\mathbb{R}^n$ a subspace U and its orthogonal complement $U^\perp$. The two subspaces U and $U^\perp$ are orthogonal, they determine each other (as $U^{\perp\perp} = U$), and together they build the entire space (as $\mathbb{R}^n = U \oplus U^\perp$). Much the same idea can be applied to the situation where the Euclidian space is replaced by an abelian category, e.g. the category of R-modules: A cotorsion pair in Mod R is a pair $(\mathcal{A},\mathcal{B})$ of classes of R-modules that are orthogonal with respect to $\mathrm{Ext}^1_R(-,-)$ (which we think of as a non-commutative inner product), they determine each other (as $\mathcal{A}^\perp = \mathcal{B}$ and $\mathcal{A} = {}^\perp\mathcal{B}$), and together they build the entire module category. Thus, to understand Mod R, it suffices to understand the two "halves" $\mathcal{A}$ and $\mathcal{B}$. People have put much effort into finding "good" cotorsion pairs. Moreover, using a result of [Hovey, 2002], people have in recent years applied cotorsion pairs to construct model structures on certain kinds of abelian categories.

In joint works with Enochs and Jørgensen I have been interested in constructing useful cotorsion pairs. E.g. in [Holm–Jørgensen, to appear] we show that every cotorsion pair $(\mathcal{A},\mathcal{B})$ in Mod R induces new cotorsion pairs $(\Phi(\mathcal{A}),\mathrm{Rep}(Q,\mathcal{B}))$ and $(\mathrm{Rep}(Q,\mathcal{A}),\Psi(\mathcal{B}))$ in the category $\mathrm{Rep}(Q,\mathrm{Mod}\,R)$ of module-valued representations of a quiver Q.

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