MY RESEARCH

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My research lies within the areas of algebraic and geometric topology. I have two main current interests: *homological stability phenomena* and *free loop spaces*, which I will try to describe with the help of a few pictures.

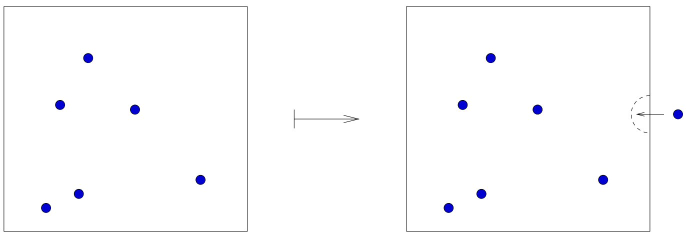
Homological stability: Certain objects come in families indexed by the natural numbers, like the general linear groups $GL_n(R)$ or the configuration space $\mathrm{Conf}_n(\mathbb{R}^2)$ of n distinct points in the plane. In the case of the general linear groups, there is a well-known group inclusion $GL_n(R) \to GL_{n+1}(R)$ that adds a 1 in the bottom corner of the matrix and fills the rest with 0's:

$$\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \mapsto \begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

so the family of general linear groups can be thought of as a sequence of groups and homomorphisms:

$$GL_1(R) \longrightarrow GL_2(R) \longrightarrow \dots \longrightarrow GL_n(R) \longrightarrow GL_{n+1}(R) \longrightarrow \dots$$

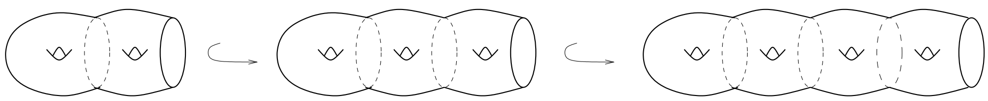
Likewise, one can include the configuration space $\mathrm{Conf}_n(\mathbb{R}^2)$ inside the configuration space $\mathrm{Conf}_{n+1}(\mathbb{R}^2)$ by "pushing an extra point in from infinity":



Hence one also has in this example a sequence, now of spaces and continuous maps between them:

$$\mathrm{Conf}_1(\mathbb{R}^2) \longrightarrow \mathrm{Conf}_2(\mathbb{R}^2) \longrightarrow \dots \longrightarrow \mathrm{Conf}_n(\mathbb{R}^2) \longrightarrow \mathrm{Conf}_{n+1}(\mathbb{R}^2) \longrightarrow \dots$$

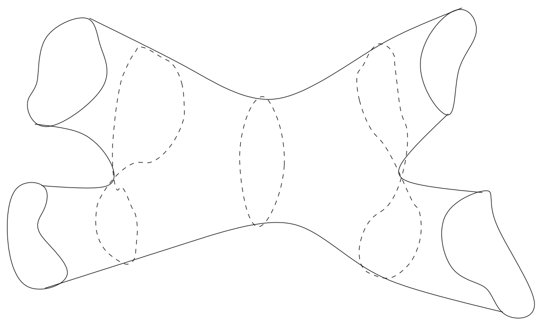
Another relevant example can be obtained from considering a surface with one boundary component, and increasing genus:



Associated to this family of surfaces is a corresponding family of diffeomorphism groups (fixing the boundary of the surface), where a diffeomorphism of the smaller surface defines a diffeomorphism of the bigger one by extending it by the identity. There is likewise an associated sequence of moduli spaces of Riemann structures on such surfaces. Common to the examples we chose is that they display a stability phenomenon, when looked at through the eyes of homology, as we will explain now.

Homology is an invariant one can associate to spaces or groups. Given a space or a group, its homology is a sequence of abelian groups H_i indexed over the natural numbers (though there are no maps in between the elements of the sequence this time!). Like most invariants, computing it can be challenging. One says that a family $X_1 \to X_2 \to \dots$ as above is *homologically stable* if for each i the homology group $H_i(X_n)$ stops changing when n is large enough. This stability phenomenon happens for many interesting families of groups (eg. the general linear groups, symmetric groups, automorphisms of free groups) and spaces (eg. certain configuration spaces, the moduli space of Riemann surfaces, the classifying spaces for certain manifold bundles). More than that, in many cases, the "stable part" of the homology, what it converges to, is often computable using methods from homotopy theory, making homological stability a very powerful tool for computations. A striking example of that was the Madsen-Weiss theorem that computed the stable part of the homology of the moduli spaces of Riemann surfaces, proving in particular a conjecture of Mumford, originating in algebraic geometry. Higher dimensional analogues of this result were obtained by Søren Galatius and Oscar Randal-Williams. Another example, which I recently obtained in joint work with Markus Szymik, was the corresponding theorem for the Higman-Thompson groups: we proved that the homology of the groups stabilizes, and computed the stable homology. In particular, we showed that the homology of Thompson's group V is trivial, answering in the positive an old question of Ken Brown.

Free loop spaces: Given a space X, one can consider the space of all loops or "strings" inside X, that is the space $\mathrm{Maps}(S^1, X)$, often called the *free loop space* of X. It is relevant to physics because of string theory, a theory where particles are replaced by small loops or paths, and to geometry because geodesics in a manifold are the critical points of the "energy" functional on the loop space. It also connects to algebra via a homology theory for algebras called Hochschild homology.



As the loop space of a space necessarily is big, giving it "structure" is essential: addition allows us to understand the natural numbers, the structure of a vector space allows us to understand $\mathbb{R}^n$. This part of my research is concerned with describing the structure of the homology of a free loop space, building in particular on *string topology*, a construction of a "loop product" defined on the homology of free loop spaces on manifolds 15 years ago by Moira Chas and Dennis Sullivan. String topology was inspired by the "string interactions" of string theory: when strings evolve in time, they meet and split, they form surfaces. String topology can nowadays be formulated as an action of the homology of the moduli space of Riemann surfaces (or its harmonic compactification) on the homology of the free loop spaces on manifolds. One of my main motivations in studying free loop spaces has actually been that they thus can serve as a representation of these moduli spaces, and I was able the identify the first infinite families of non-trivial homology classes in the Harmonic compactification of moduli space using string topology.