Real spherical spaces and Plancherel theory

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Representation theory is the all-embracing topic of my research. Most often it involves a group G, a subgroup H, and the homogeneous space $Z = G/H$. The representation theory associated with Z concerns the *regular* representation L of G, which is defined on functions on Z by letting $L_g f(z) = f(g^{-1}z)$. It is the aim of Plancherel theory to decompose this representation into irreducible representations. In order to do this in useful terms we need to impose further assumptions on Z and on the space of functions considered.

Let us assume G is locally compact and that there is a G-invariant measure on Z. Locally compact groups always carry an invariant measure (Haar measure), but their quotients need not. However, when such a measure exists it is unique up to normalization by a scalar, and it makes sense to form the space $L^2(Z)$. It is the regular representation on this space that we would like to decompose into irreducibles. By a general theorem, the *abstract Plancherel theorem*, this is possible under very general assumptions on G. One problem to which I (among many others) have devoted some effort is to obtain this Plancherel decomposition explicitly in given circumstances. In particular, one wants to know which representations occur in the decomposition, and one wants a reasonably explicit formula for the associated Fourier transform, which maps an L^2-function equivariantly into vectors in the representation spaces.

An instructive example is obtained when G is compact. Here the theorem of Peter and Weyl implies the following L^2-version of Frobenius reciprocity

$$L^2(Z) \simeq \widehat{\bigoplus}_{[\pi]\in\widehat{G}} \dim(\pi)\,(V_\pi^H)^* \otimes V_\pi$$

where the sum extends over the set $\widehat{G}$ of all equivalence classes of irreducible representations (π, V_π) of G, and where V_π^H denotes the subspace of H-fixed vectors in V_π. In particular, only representations from $\widehat{G}_H := \{[\pi] \mid V_\pi^H \neq 0\}$ occur. The scalar multiple in front of the tensor product signifies that in the direct sum the square norm of a vector is defined with $\dim(\pi)$ times counting measure (over $\widehat{G}_H$), together with the square of the tensor product norm. This multiple of counting measure is called the *Plancherel measure* associated to Z. The associated Fourier transform $f \mapsto \mathcal{F}f$ is given by the vector valued integral

$$\mathcal{F}f(\pi) := \left(\eta \mapsto \int_Z f(z)\pi(z)\eta\,dz\right) \in \mathrm{Hom}(V_\pi^H, V_\pi) \simeq (V_\pi^H)^* \otimes V_\pi,$$

and the *Plancherel formula* asserts that this is an isometry.

For a non-compact space Z**,** the decomposition of $L^2(Z)$ will generally not consist of a direct sum as above. This happens already in the Plancherel theory for $L^2(\mathbb{R})$, which amounts to the theory of Fourier integrals. In general there will be both a discrete spectrum and a continuous spectrum. This is for example seen in the important case where G is a non-compact reductive Lie group and H is trivial. For this case Harish-Chandra determined the Plancherel decomposition of $L^2(G)$ in a long series of papers, the last one of which appeared in 1976. Of particular interest is the *discrete series*, which consists of all the irreducible representations that embed into $L^2(G)$, thus contributing the discrete spectrum. Every other occurring irreducible representation is described explicitly by means of a parabolic subgroup P of G together with some data from which it is induced. There are finitely many such subgroups P up to conjugation, and each conjugacy class contributes a series of representations, called *principal series.* Harish-Chandra's Plancherel formula thus decomposes $L^2(G)$ into a sum of discrete series and various principal series, each of which decomposes continuously into irreducibles with a Plancherel measure that can be described in explicit terms.

An important ingredient in the theory is *Harish-Chandra's criterion*:

The discrete series for G is not empty $\Leftrightarrow$ G contains a compact maximal torus

Symmetric spaces give a profitable framework for Plancherel theory. Here we deal with the same reductive groups G as before, but with a non-trivial subgroup H, assumed to be the fixed points of an involution of G. By regarding G as a homogeneous space for the two-sided action of $G \times G$ one can view the previous case of $Z = G$ as a special case.

The presence of an involution fixing H gives rise to a rich and well understood geometric structure on the pseudo-Riemannian symmetric space $Z = G/H$. Based on that a Plancherel decomposition was determined by P. Delorme, and with a different proof also by E. van den Ban and myself. An important ingredient was a generalization of Harish-Chandra's criterion for the discrete series, due to Flensted-Jensen ($\Leftarrow$) and Oshima-Matsuki ($\Rightarrow$).

Real spherical spaces is a recent and exciting development. This is a more general class of homogeneous spaces, for which a geometric structure theory has been developed lately.

By definition, a homogeneous space $Z = G/H$ of a real reductive Lie group is *real spherical* if a minimal parabolic subgroup of G admits an open orbit on it (which is the case for symmetric spaces). In a recent preprint together with Delorme, Knop, and Krötz, we develop the ingredients of a Plancherel theory for such spaces. By using a compactification of Z we find a finite number of deformations $Z_I = G/H_I$ of Z, called its *boundary degenerations*, indexed by subsets of a certain finite set associated with the fine geometry of Z. Their presence on the boundary of Z implies a relation to L^2-growth estimates for functions on Z.

The boundary degenerations are homogeneous spaces with a structure that allows the construction of irreducible representations of G by induction from the normalizer of H_I in G. The various series of representations induced from these normalizers comprise all representations that occur in the Plancherel decomposition for Z, except for the discrete series. We thus obtain a collection of equivariant isometries, so-called *Bernstein morphisms*, from the induced series into $L^2(Z)$, and we prove that their images together with the discrete series span the entire space. What remains is to determine the overlaps between these images.

For the discrete series we obtain as a corollary a generalization of Flensted-Jensen's arrow $\Leftarrow$, which had been conjectured by Krötz, Kuit, Opdam, and myself. What also remains is then to prove (or disprove) the converse arrow $\Rightarrow$.

It is remarkable that a main source of inspiration for this research on real spherical spaces is p-adic: In a 370 page paper from 2017, *Periods and harmonic analysis on spherical varieties,* Sakellaridis and Venkatesh develop a Plancherel theory for spherical varieties over non-Archimedian local fields.