TENSOR-TRIANGULAR GEOMETRY AND EQUIVARIANT STABLE HOMOTOPY THEORY COPENHAGEN, MARCH 12-16

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Lecture plan

Lecture 1. Tensor-triangulated categories (Paul)

Lecture 2. Equivariant homotopy theory (Justin)

Lecture 3. The spectrum and classification of tt-ideals (Paul)

Lecture 4. Nilpotence and descent in equivariant stable homotopy theory (Justin)

Lecture 5. Separable extensions (Paul)

Lecture 6. $\mathcal{F}$-nilpotence and applications (Justin)

Lecture 7. Faithful descent and applications (Paul)

Lecture 8. TT-ideals in genuine A-spectra. (Justin)

Lecture 9. Residue fields and nilpotence theorem (Paul)

Lecture 10. Verifying $\mathcal{F}$-nilpotence (Justin)

Abstracts

Abstract 1. *Tensor-triangulated categories:* We review the definition and examples of tensor-triangulated categories in algebraic geometry, representation theory and beyond. We also discuss the tensor structure, rigidity, compactness, ring objects and tt-ideals.

Abstract 2. *Equivariant homotopy theory:* In this lecture we will give a quick introduction to the formal aspects of equivariant homotopy theory including stable equivariant homotopy theory with respect to a finite group G. We will discuss classifying spaces for families of subgroups and the zoo of fixed point functors. We will also introduce some of our star players: the sphere spectrum, equivariant topological K-theory, Mackey functors, and the 'Borel-complete' theories.

Abstract 3. *The spectrum and classification of tt-ideals:* We discuss the spectrum of a tensor-triangulated category and relate it to the classification of tt-ideals. We give several examples. We discuss first results one can prove using the spectrum and rigidity.

Abstract 4. *Nilpotence and descent in equivariant stable homotopy theory:* We consider a version of descent in equivariant homotopy theory and the associated homotopy (co)limit spectral sequences associated to a family of subgroups, and the notion of a quickly converging tower.

Abstract 5. *Separable extensions:* We explain how separable commutative ring objects generalize localization, in a way reminiscent of how the étale topology generalizes the Zariski topology. We give examples in equivariant settings. We discuss the effect of such separable extensions on the spectrum.

Abstract 6. *$\mathcal{F}$-nilpotence and applications:* We will introduce the tt-ideal of $\mathcal{F}$-nilpotent spectra and give two equivalent characterizations of these spectra. The first of which is a set of nilpotence criteria which is easy to check in practice. The second characterization is computational and leads to generalized Artin induction theorems as well as generalizations of Quillen's $\mathcal{F}$-isomorphism theorem.

Abstract 7. *Faithful descent and applications:* An older, special case of descent-up-to-nilpotence is plain faithful descent, i.e. the case of a ring object in a tt-category which is faithful. We then discuss in some detail the case of restriction to subgroups in modular representation theory, with endotrivial modules in mind. We show how these questions relate to line bundles on the Brown-Quillen complex and to results of Grodal.

Abstract 8. *TT-ideals in genuine A-spectra:* After the work of Balmer and Sanders on the Balmer spectrum of genuine G-spectra, the last remaining task to classify the tt-ideals in genuine G-spectra is to determine the inclusions between the prime ideals. This is essentially a question about how the chromatic type of the geometric H-fixed points of a finite G-spectrum vary as we change H and was solved by Balmer and Sanders in the case that $|G|$ is square-free.

We can solve this provided every subgroup of G is normal (e.g., when G is abelian), by bounding the ranges of the possible chromatic types. A result of Arone, Dwyer, and Lesh tells us that there are finite G-spectra which provide one direction of the bound. Using a recent generalization of Kuhn's blue-shift theorem about Tate cohomology we obtain the other bound.

Abstract 9. *Residue fields and nilpotence theorem:* We explain the problem of constructing residue fields for local tt-categories. We attack this question via the abelian Freyd envelope and show that the homological residue fields that are obtained this way detect tensor-nilpotence of maps.

Abstract 10. *Verifying $\mathcal{F}$-nilpotence:* We will demonstrate that the nilpotence criteria for $\mathcal{F}$-nilpotence is indeed 'easy to check' by actually checking it in a sequence of a examples including equivariant topological K-theory, and complex-oriented Borel equivariant cohomology theories.

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