If A is of p-rk k then

$$A \twoheadrightarrow C_p^{\times k} \longrightarrow C_p^{\times N}$$

& we can pull back ADLM spaces to show we can't lift our inclusions further up the tower.

Lecture 9 – Applications (Balmer)

Thm: Suppose $\mathcal{K}$ tt-cat. rigid & idem. complete. Suppose $a \in \mathcal{K}$ is s.t. $\mathrm{supp}(a) = Z_1 \amalg Z_2$ (Z_i closed & disjoint). Then $\exists\, a_1, a_2 \in \mathcal{K}$ s.t. $a \cong a_1 \oplus a_2$ and $\mathrm{supp}\, a_i = Z_i$

Proof: Let $J = \{ a_1 \oplus a_2 \mid \mathrm{supp}\, a_i \subseteq Z_i \} \subseteq \mathcal{K}$

J is a $\otimes$-ideal ✓

triangulated, thick } need: **Lemma**: if $\mathrm{supp}(a_1) \cap \mathrm{supp}(a_2) = \emptyset$ then $\hom_{\mathcal{K}}(a_1, a_2) = 0$

Proof: $\mathrm{supp}(a_1^\vee \otimes a_2) = \emptyset$

$\Rightarrow a_1^\vee \otimes a_2 = 0$

$\Rightarrow \hom(a_1, a_2) = \hom(\mathbb{1}, a_1^\vee \otimes a_2) = 0$ □

Now every map in J is of the form

$$a_1 \oplus a_2 \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}} b_1 \oplus b_2$$

(by the lemma)

So J is a tt-ideal, and $J = \mathcal{K}_{Z_1} \oplus \mathcal{K}_{Z_2}$ ($= \mathcal{K}_{Z_1} \vee \mathcal{K}_{Z_2}$) poset join

(recall $\mathcal{K}_Y = \{ x \in \mathcal{K} \mid \mathrm{supp}\, x \subseteq Y \}$)

$\xRightarrow{\text{classification}}$ $J = \mathcal{K}_{Z_1 \cup Z_2}$ (as $Z_1 \cup Z_2 = Z_1 \vee Z_2$)

$\therefore\ a \in \mathcal{K}_{Z_1 \cup Z_2} \Rightarrow a \in \mathcal{K}_{Z_1} \oplus \mathcal{K}_{Z_2}$ □

Filter $\mathcal{K}$ by dim^n of supp:

$$\mathcal{K}^{(p)} := \{ x \in \mathcal{K} \mid \mathrm{codim}(\mathrm{supp}(x)) \geq p \}$$

$\cup$

$\mathcal{K}^{(p+1)}$

Krull dim^n

$$\left(\frac{K^{(p)}}{K^{(p+1)}}\right)^{\natural} \xrightarrow{\sim} \coprod_{\substack{x\in Spc(K)\\ ht(x)=p}} \underbrace{\overbrace{(K_x)}^{\text{local cat at } x}_{\{x\}}}_{\substack{\text{obs supported on } \{x\}\\ \text{"smallest subcat of } K_x\text{"}}}$$

Gluing:

Notation: $K(\mathcal{U}) := \left(K/K_Z\right)^{\natural}$ $\left(\begin{array}{l} Z := Spc(K)\setminus\mathcal{U} \\ \mathcal{U} \text{ q.c. open}\end{array}\right)$

$Spc(K) = \mathcal{U}_1 \cup \mathcal{U}_2$

$$\Rightarrow \quad \begin{array}{ccc} K & \longrightarrow & K(\mathcal{U}_1) \\ \downarrow & & \downarrow \\ K(\mathcal{U}_2) & \longrightarrow & K(\mathcal{U}_1\cap\mathcal{U}_2) \end{array}$$

Warning: in general $\exists f \in K$ s.t. $f|_{\mathcal{U}_i} = 0$ for $i=1,2$ but $f \neq 0$.

Example: X scheme, $\xi_1 \rightarrowtail \xi_2 \twoheadrightarrow \xi_3$ ex. seq. of v. bundles

$\rightsquigarrow$ ex. Δ $\quad \xi_1 \longrightarrow \xi_2 \longrightarrow \xi_3 \xrightarrow{w} \Sigma\xi_1$ in $D(X)$

$w = 0 \iff$ the seq. is split exact.

In particular, $w \mapsto 0$ under $D(X) \to D(\mathcal{U})$ $\forall\, \mathcal{U} \subseteq X$ affine.

e.g. $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$.

For 2 opens: $\exists$ gluing of objs up to non-unique iso (non-uniqueness exactly corresponds to the "noise" from the above warning)

For 3 opens: $\exists$ gluing, not even unique up to iso (due to 2-open non-uniqueness)

For ≥ 4: no gluing, a priori.

M-V exact sequence.

$$\to \mathrm{Hom}_{\mathcal{U}_1\cap\mathcal{U}_2}(\Sigma a, b)$$

$$\hookrightarrow \mathrm{Hom}_{\mathcal{K}}(a,b) \longrightarrow \mathrm{Hom}_{\mathcal{K}(\mathcal{U}_1)}(a,b) \oplus \mathrm{Hom}_{\mathcal{U}_2}(a,b) \to \mathrm{Hom}_{\mathcal{U}_1\cap\mathcal{U}_2}(a,b)$$

$$\hookrightarrow \mathrm{Hom}_{\mathcal{K}}(a, \Sigma b) \to \dots$$

$$\mathbb{G}_m(\mathcal{K}) := \mathrm{Aut}_{\mathcal{K}}(\mathbb{1}), \quad \mathcal{O}_{\mathcal{K}}(\mathcal{U}) := \mathrm{End}_{\mathcal{K}(\mathcal{U})}(\mathbb{1})$$

$$\dots \to \mathrm{Hom}_{\mathcal{U}_1\cap\mathcal{U}_2}(\mathbb{1}, \Sigma\mathbb{1}) \to \mathbb{G}_m(\mathcal{K}) \to \mathbb{G}_m(\mathcal{U}_1) \oplus \mathbb{G}_m(\mathcal{U}_2) \to \mathbb{G}_m(\mathcal{U}_1\cap\mathcal{U}_2)$$

$$\hookrightarrow \mathrm{Pic}(\mathcal{K}) \to \mathrm{Pic}(\mathcal{U}_1) \oplus \mathrm{Pic}(\mathcal{U}_2) \to \mathrm{Pic}(\mathcal{U}_1\cap\mathcal{U}_2)$$

Good news: f locally zero $\Rightarrow f^{\otimes n} = 0$

Example: If $\mathcal{K}$ is an $\mathbb{F}_p$-cat, can play with $(-)^{p^r}$, $r \gg 0$

Thm: If $\mathcal{K}$ $\mathbb{F}_p$-linear, there exists

$$\mathrm{Pic}(\mathrm{Spc}(\mathcal{K})) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{p}] \hookrightarrow \mathrm{Pic}(\mathcal{K}) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{p}]$$

[since $(1+h)^{p^r} = 1 + h^{p^r}$, this inverting p kills the "noise"]

Example: $\mathcal{K} = \mathrm{stmod}_{kG}$, G finite gp. The above induces isom.

$$\mathrm{Pic}(\mathcal{V}_G) \otimes \mathbb{Q} \simeq T_k(G) \otimes \mathbb{Q}$$

$$\mathcal{V}_G := \mathrm{Spc}(\mathcal{K}) = \mathrm{Proj}\, H^*(G,k)$$

$$T_k(G) := \mathrm{Pic}(\mathrm{stmod}(kG)) = \{[M]_{\simeq} \mid M \otimes M^\vee \simeq k \oplus (\text{proj.})\}$$

"endotrivial modules"

Carlson-Thévenaz: $T_k(P)$ for any p-grp

Q: what is $\ker(T_k(G) \to T_k(P))$ where G finite gp & P a p-Sylow subgp?

$\mathrm{Res}^G_P : \mathcal{K}(G) \longrightarrow \mathcal{K}(P)$, $\mathcal{K}(G) = \mathrm{stmod}\, kG$
is fully faithful, ext. of scalars wrt $k[G/P] = A^G_P$

Classical descent ⇒ can describe $\mathcal{K}(G)$ as objects in $\mathcal{K}(P)$ + descent data

$M \in A\text{-Mod}_{\mathcal{K}}$ $\quad \gamma: A\otimes M \xrightarrow{\sim} M \otimes A$ as $A\otimes A$-mod.
$\mathcal{K}(P\cap {}^gP)$

+ cocycle cond

$$\begin{tikzcd} A\otimes A\otimes M \arrow[rr, "\gamma_1", "\sim"'] \arrow[dr, "\gamma_2"', "\wr"] & \arrow[d, phantom, "\circlearrowleft"] & A\otimes M\otimes A \arrow[dl, "\gamma_3", "\wr"'] \\ & M\otimes A\otimes A & \end{tikzcd}$$

as $A^{\otimes 3}$-mod
$\mathcal{K}(P\cap {}^gP\cap {}^{gh}P)$

Thm: $T_k(G,P) \simeq A_k(G,P)$
where $A_k(G,P)$ = "weak homomorphisms"
:= fns $u: G \to k^\times$ s.t.
1) $u(h)=1 \quad \forall h\in P$
2) $u(g)=1 \quad \forall g$ s.t. $P\cap {}^gP = 1$
3) $u(g_2g_1) = u(g_2)u(g_1) \quad \forall g_1,g_2$ s.t. $P\cap {}^{g_1}P\cap {}^{g_2g_1}P \neq 1$

⇝ C+T made some conjectures
⇝ J. Grodal : $\mathcal{O}_p^*(G) \longrightarrow k^\times$
$\Rightarrow \quad A_k(G,P) = H^1(\mathcal{O}_p^*(G); k^\times)$
Solved the conjectures

Lecture 10 - Verifying $\mathcal{F}$-nilpotence (Noel)

Let $R_G \in CAlg(Sp_G)$ $\quad \mathcal{F}$ family of subgps

$$\mathrm{Mod}_{Sp_G}(R_G) \xrightarrow[\otimes]{\mathrm{Res}_{\mathcal{F}}} \lim_{\overleftarrow{\mathcal{O}(G)^{op}_{\mathcal{F}}}} \mathrm{Mod}_{Sp_G}(F(G/H_+, R_G))$$

$$\text{IS BDS}$$

$$\lim_{\overleftarrow{\mathcal{O}(G)^{op}_{\mathcal{F}}}} \mathrm{Mod}_{Sp_H}(\mathrm{Res}^G_H R_G)$$

$$\text{IS}$$

$$\mathrm{Mod}_{Sp_G}(R_G)[\mathcal{F}\text{-complete}]$$

If R_G is $\mathcal{F}$-nilpotent then it is $\mathcal{F}$-complete; moreover all of its modules are $\mathcal{F}$-complete

[$X \in Sp_G$ is $\mathcal{F}$-complete $\Leftrightarrow X \xrightarrow{\sim} F(E\mathcal{F}_+, X)$]

Recall: $\mathcal{F}$-nilpotent G-spectra is the full subcategory spanned by thick $\otimes$-ideals gen. by $\{G/H_+\}_{H \in \mathcal{F}}$.

We have a map $\boxed{S := S_G}$

$$\begin{tikzcd} S \arrow[r] \arrow[rr, bend right, "= {[G/H]} \in \pi_0^G S = A(G)"'] & F(G/H_+, S) \simeq G/H_+ \wedge S \arrow[r] & S \end{tikzcd}$$

Fix $G_p \leq G$ a p-Sylow subgroup

$$\begin{tikzcd} S \arrow[r] \arrow[rr, bend left, "{[G/G_p]}"] & G/G_{p+} \arrow[r] \arrow[d, "\underline{(\,)_{(p)}} = F(EG_+, (-)_{(p)})"] & S \\ \underline{S_{(p)}} \arrow[r] & \underline{G/G_{p+(p)}} \arrow[r] & \underline{S_{(p)}} \end{tikzcd}$$

$$\underline{G/G_{p+(p)}} \cong G/G_{p+} \wedge \underline{S_{(p)}}$$

This restricts under Res_e^G to a self-map of order $|G/G_p|$ & this is a unit in $\pi_0^G \underline{S_{(p)}}$ so the composite is an equivalence after composing w/ mult by a unit is a retract.

$\Rightarrow \underline{S_{(p)}}$ is nilpotent for the family $All_{(p)} = \{p \text{ subgps of } G\}$

$\Rightarrow \forall$ p-local spectrum X, $\underline{X}$ is $All_{(p)}$-nilpotent

~~Add~~ Example: $\underline{MU}$. Fix $G \hookrightarrow U(N)$

$F = U(N)/T^N \circlearrowleft G$, finite G-CW cx w/ isotropy in the abelian subgps of G.

Classical computation: the map $F \to *$ induces

$$\begin{array}{ccc} MU^*(BG) & \xrightarrow{\text{ring map}} & MU^*(EG \times_G F) \\ \| & & \cong \\ \underline{\pi}_*^G \underline{MU} & & \underline{\pi}_*^G \underline{MU} \wedge D(F) \end{array}$$

Moreover the target is a finite nonzero free module of rank k over the source. This is also true after restricting to any subgroup of G.

$$\bigvee_{i \in I} \Sigma^{2i} \underline{MU} \xrightarrow[\text{G-equiv}]{\sim} \underline{MU} \wedge D(F)$$

$\Rightarrow$ $\underline{MU}_G$ is a retract of ~~$\underline{MU} \wedge D(F)$~~ $\underline{MU} \wedge D(F)$

in thick subcat gen. by $\{G/A_+\}$, A abelian.

$\Rightarrow$ $\underline{MU} \wedge D(F)$ is Ab-nilpotent

abelian subgps of G

$\Rightarrow$ $\underline{MU}_G$ ——— " ———

An up-to-homotopy ring spectrum E is called $\mathbb{C}$-orientable if $\exists$ ring map $MU \to E$.

$\Rightarrow$ $\underline{E}$ is an $\underline{MU}$-module.

If $E \simeq E_{(p)}$ then $\underline{E}_{(p)}$ is $All_{(p)} \cap Ab$-nilpotent.

$\Leftrightarrow$ $\Phi^H \underline{E}_{(p)} \simeq 0$ $\forall H \leq G$ non-abelian.

$\Rightarrow$ HKR induction theorems.

Hopkins-Kuhn-Ravenel

Recall that $X \in \mathcal{F}^e$-nilpotent $\cap \, Sp_G$

$\Leftrightarrow \forall H \leq G,\ H \notin \mathcal{F}^e$: $\mathrm{Res}^G_H X$ is nilpotent for the family of proper subgroups of H.

$\Leftrightarrow \forall H \notin \mathcal{F}^e$: $\cancel{\Phi^H X \simeq 0}$.

$$\Phi^H \mathrm{End}(X) \simeq 0$$
$$\shortparallel$$
$$\left(\mathrm{End}(X)[e_{\tilde{\rho}_H}^{-1}]\right)^H$$

If R_G is a htpy comm. G-ring spectrum and $\mathcal{J}$ is a collection of orthogonal rep'ns of G that is closed under $\oplus$ (and they should exhaust a complete G-universe)

We say R_G has mult. Thom classes if we have isos $R_H^*(S^{|V|}) \cong R_H^*(S^V)$ of R_H^*-modules natural in H.

$\Rightarrow F(S^{|V|}, R_G) \xrightarrow{\sim} F(S^V, R_G)$ of R_G-modules.

$$\begin{tikzcd}
R_G & S^{V-|V|} \arrow[l, "\mu_V"'] \arrow[d] \\
R_G \wedge R_G \arrow[u, "\mu"] & S^{V-|V|} \wedge R_G \arrow[l] \arrow[ul, dotted, "\sim"]
\end{tikzcd}$$

Moreover $\mu_{V+W} = \mu_V \cdot \mu_W$

$$\begin{tikzcd}
S^{-|V|} \wedge S^0 \arrow[r, "e_V"] \arrow[rr, bend right, "\chi(V)"'] & S^V \wedge S^{-|V|} \arrow[r, "\mu_V"] & R_G
\end{tikzcd}$$

$\cancel{\chi(V)}$

Satisfies $\chi(V+W) = \chi(V)\chi(W)$

$\Rightarrow$ e_V is nilpotent in $\pi_\star^G R$
iff $\chi(V)$ is nilpotent in $\pi_\star^G R$.

If R_G has multiplicative Thom classes for $km\tilde{\rho}_G$ $\forall k \geq 0$, some fixed m (e.g. R_G $\mathbb{C}$-oriented, $m=2$
R_G is $Spin(8k)$ oriented, $m=8$)

then $\Phi^G R_G = 0 \Leftrightarrow \mathcal{N}(m\tilde{\rho}_G)$ is nilpotent

$\Leftrightarrow R_G$ is p-nilpotent

$\Rightarrow \forall x \in \pi_*^G R_G$ which restricts to a nilpotent element on all proper subgps, x is nilpotent

$\Rightarrow \mathcal{N}(m\tilde{\rho}_G)$ is nilpotent

(so all $\Rightarrow$ are $\Leftrightarrow$).

$\forall G$ abelian p-gp. not elem. ab.

$\Rightarrow \underline{H\mathbb{F}_p}$ is $\mathcal{E}_{(p)}$-nilpotent iff $\forall x \in H^*(BG; \mathbb{F}_p)$ which restricts to zero on every proper subgp of G is nilpotent.

Suppose A is an abelian p-grp gen'd by N elements (and not fewer) and fix $\Delta = C_p^{\times N} \hookrightarrow A \hookrightarrow \mathbb{T}^N$

$$H^*(B\mathbb{T}^N; \mathbb{F}_p) \xrightarrow{i^*} H^*(BA, \mathbb{F}_p) \xrightarrow{f^*} H^*(B\Delta; \mathbb{F}_p)$$

$$H^*(B\mathbb{T}^N; \mathbb{F}_p) = \mathbb{F}_p[t_1, \dots, t_N], \quad |t_i| = 2$$

the p^{th} power of any class in $H^*(BA; \mathbb{F}_p)$ is in the image of i^*, hence the p^{th} power is nilpotent $\Leftrightarrow$ it is zero.

$\Rightarrow$ any class that restricts to 0 under f^* is nilpotent

$\Rightarrow \underline{H\mathbb{F}_p}$ is $\mathcal{E}_{(p)}$-nilpotent.

$\underline{E_N}$ is nilpotent for abelian p-gps of rank $\leq N$

<u>Smash product thm of Hopkins-Ravenel</u>:

$S_{E_N} =: L_N S \longrightarrow E_N$ and $L_N S$ is in the thick $\otimes$-ideal gen'd by E_N.

$\uparrow$ Bousfield loc.

$\Rightarrow \underline{L_N S}$ is nilpotent for same family as $\underline{E}_N$.