THREE PERSECTIVES ON DELIGNE COHOMOLOGY

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Contents

1. Talk 1: Deligne cohomology

Deligne cohomology is a cohomology theory for complex manifolds which refines the usual singular/sheaf cohomology $\mathrm{H}^*(M;\mathbb{Z})$ by including some differential form data.

1.1. Motivation.

Let V be a holomorphic vector bundle over a complex manifold M. Then we get a complex topological vector bundle on the topological space M, hence a Chern class $c_p(V) \in \mathrm{H}^{2p}(M;\mathbb{Z})$.

We want refined coefficients $\mathbb{Z}(p)_{\mathrm{D}}$ which maps to $\mathbb{Z}$ such that $c_p(V) \in \mathrm{H}^{2p}(M;\mathbb{Z})$ functorially (with respect to pullback of vector bundles) lifts to $\mathrm{H}^{2p}(M;\mathbb{Z}(p)_{\mathrm{D}})$.

Remark 1.1. Why do the coefficients depend on p? Note that $c_1(\mathcal{L}) \in \mathrm{H}^2(M;\mathbb{Z})$, where $\mathbb{Z}$ should be identified with $\mathrm{H}_1(\mathbb{C}^\times;\mathbb{Z}) \cong 2\pi i\mathbb{Z} \subseteq \mathbb{C}$.

Similarly, we should have $c_p(V) \in \mathrm{H}^{2p}(M;(2\pi i)^p\mathbb{Z})$.

To define $\mathbb{Z}(p)_{\mathrm{D}}$, let us look at first Chern classes:

$$c_1(V) = c_1(\det(V)).$$

One description is the following: The short exact sequence

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 1.$$

gives a map $\mathrm{H}^1(M;\mathcal{O}^\times) \to \mathrm{H}^2(M;2\pi i\mathbb{Z})$. This suggests setting

$$\mathbb{Z}(1)_{\mathrm{D}} = \mathcal{O}^\times[-1].$$

Reinterpretation: We have a homotopy pullback (because the cofibers on both horizontal maps identify with $\mathcal{O}$):

$$\begin{tikzcd} \mathcal{O}^\times[-1] \arrow[r, "\partial"] \arrow[d, "\mathrm{dlog}(f)=\frac{\mathrm{d}f}{f}"'] & 2\pi i\mathbb{Z} \arrow[d] \\ {[0 \to \Omega^1 \to \Omega^2 \to \cdots]} \arrow[r] & {[\Omega^0 \to \Omega^1 \to \cdots]} \end{tikzcd}$$

where $\Omega_{\mathrm{dR}} \coloneqq (\Omega^\bullet, d)$ is the holomorphic de Rham complex.

Definition 1.2. For $p \in \mathbb{Z}$, $p \geq 0$, define

$$\begin{tikzcd} \mathbb{Z}(p)_{\mathrm{D}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (2\pi i)^p\mathbb{Z} \arrow[d] \\ F^p\Omega_{\mathrm{dR}} = [0 \to \cdots \to 0 \to \Omega^p \to \Omega^{p+1} \to \cdots] \arrow[r] & {[\Omega^0 \to \Omega^1 \to \cdots] = \Omega_{\mathrm{dR}},} \end{tikzcd}$$

where on the left Ω^p sits in homological degree $-p$.

Example 1.3.
 (i) $\mathbb{Z}(0)_{\mathrm{D}} = \mathbb{Z}$;
 (ii) $\mathbb{Z}(1)_{\mathrm{D}} = \mathcal{O}^\times[-1]$, which implies that there is a tautological Chern class $c_1(\mathcal{L}) \in \mathrm{H}^2(M; \mathbb{Z}(1)_{\mathrm{D}})$.
 (iii) $\bigoplus_p \mathbb{Z}(p)_{\mathrm{D}}$ is a graded commutative ring, i.e., there is a map $\mathbb{Z}(p)_{\mathrm{D}} \otimes \mathbb{Z}(q)_{\mathrm{D}} \to \mathbb{Z}(p+q)_{\mathrm{D}}$.
 (iv) Pullback functoriality: for each $M \to N$ there is a map $\mathrm{H}^*(N; \mathbb{Z}(p)_{\mathrm{D}}) \to \mathrm{H}^*(M; \mathbb{Z}(p)_{\mathrm{D}})$.
 (v) Projective bundle formula: Let $V \to M$ be a vector bundle of dimension d and consider its projectivization $\mathbb{P}(V) \to M$. Then

$$\bigoplus_p \mathrm{R}\Gamma(\mathbb{P}(V); \mathbb{Z}(p)_{\mathrm{D}})$$

is graded free of rank d over $\bigoplus_p \mathrm{R}\Gamma(M; \mathbb{Z}(p)_{\mathrm{D}})$ on $1, c_1(\mathcal{O}(1)), \ldots, c_1(\mathcal{O}(1))^{d-1}$.

By Grothendieck, we can expand $c_1(\mathcal{O}(1))^d$ in terms of previous powers, and the coefficients define the higher Chern classes $c_p(V) \in \mathrm{H}^{2p}(M; \mathbb{Z}(p)_{\mathrm{D}})$.

Example 1.4. If $p \leq 0$, then $\mathbb{Z}(p)_{\mathrm{D}} = (2\pi i)^p\mathbb{Z}$, which is "purely topological".
If $p > \dim M$, then $\mathbb{Z}(p)_{\mathrm{D}} = \mathbb{C}/(2\pi i)^p\mathbb{Z}[-1]$.

Remark 1.5. Let M be Stein (i.e., a closed submanifold $M \hookrightarrow \mathbb{C}^N$). Then Ω^i is acyclic and hence $\mathrm{H}^p(F^p\Omega_{\mathrm{dR}}) = \Omega^p_{\mathrm{cl}}$ is the space of holomorphic closed p-forms (these are huge vector spaces!).

If M is compact, then $\dim_{\mathbb{C}}(\bigoplus_{i,p} \mathrm{H}^i(M; \Omega^p)) < \infty$ and hence the Deligne cohomlogy groups are always built out of $\mathbb{Z}$'s and $\mathbb{C}$'s by extensions and quotients.

If M is compact Kähler (e.g., a smooth projective variety over $\mathbb{C}$), then the map

$$\mathrm{H}^*(M; F^p\Omega_{\mathrm{dR}}) \to \mathrm{H}^*(M; \Omega_{\mathrm{dR}}) \simeq \mathrm{H}^*(M; \mathbb{C})$$

is *injective*. The image is $F^p\mathrm{H}^*(M; \mathbb{C})$ in the Hodge filtration. Thus, we have a short exact sequence

$$0 \to \frac{\mathrm{H}^{i-1}(M; \mathbb{C})}{F^p\mathrm{H}^{i-1}(M; \mathbb{C}) + \mathrm{H}^{i-1}(M; (2\pi i)^{p-1}\mathbb{Q})} \to \mathrm{H}^i(M; \mathbb{Q}(p)_{\mathrm{D}}) \to F^p\mathrm{H}^i(M; \mathbb{C}) \cap \mathrm{H}^i(M; (2\pi i)^p\mathbb{Q}) \to 0,$$

which we view as an extension of something "discrete" by something "continuous".

When $i = 2p$, the left hand side is $J^p(M)_{\mathbb{Q}}$, that is, Griffith's intermediate Jacobian; for $p = 1$ this is the usual Jacobian.

1.2. **Applications.**

(i) One application, the intermediate Jacobians $J^p(M)$, were already mentioned.

(ii) Secondary characteristic classes of flat bundles

$$\begin{tikzcd} c_p(V) \in \mathrm{H}^{2p-1}(B\,\mathrm{GL}_n(\mathbb{C})^\delta, \mathbb{C}/(2\pi i)^p\mathbb{Z}), \arrow[d, "\partial"] & & p>0 \\ \mathrm{H}^{2p}(B\,\mathrm{GL}_n(\mathbb{C})^\delta, (2\pi i)^p\mathbb{Z}), \end{tikzcd}$$

also called the Chern–Simons invariants.

(iii) **Arithmetic:** Let $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ be a regular proper scheme over $\mathbb{Z}$. Then $\mathcal{X}$ should not be thought of as compact, because $\mathrm{Spec}(\mathbb{Z})$ is not compact ($\mathrm{Spec}(\mathbb{Z})$ corresponds to the affine line $\mathrm{Spec}(\mathbb{F}_p[T]) = \mathbb{A}^1_{\mathbb{F}_p}$.

There is a small neighborhood around $\infty \in \mathbb{P}^1_{\mathbb{F}_p}$ corresponding to $\mathbb{F}_p[T] \to \mathbb{F}_p((T^{-1}))$ (which which should be thought of as corresponding to the inclusion $\{0\} \to \mathbb{R}$).

On $\mathcal{X}$, we should consider not just cohomology, but "compactly supported cohomology", namely

$$\mathrm{fib}\big(\mathrm{R}\Gamma(\mathcal{X}) \to \mathrm{R}\Gamma(\mathcal{X}(\mathbb{C})/C_2)\big).$$

The idea of Arakelov theory is the following: If "cohomology" means motivic cohomology, then $\mathrm{R}\Gamma(\mathcal{X}(\mathbb{C}))$ should be Deligne cohomology.

Example 1.6. Consider the motivic cohomology

$$\mathrm{H}^i_{\mathrm{M}}(\mathrm{Spec}(\mathbb{Z}); \mathbb{Q}(p)) \to \mathrm{H}^i(*_{\mathbb{C}}; \mathbb{Q}(p)_{\mathrm{D}})^{C_2}.$$

In weights $p > 1$, (by Borel) the left hand side is non-zero if and only if $i = 1$ and p is odd, in which case it is a one-dimensional $\mathbb{Q}$-vector space. The right hand side is non-zero if and only if $i = 1$ and p is odd, in which case it is isomorphic to $\mathbb{C}/(2\pi i)^p\mathbb{Q} \xrightarrow{\mathrm{Re}} \mathbb{R}$. The image of the induced map $\mathbb{Q} \to \mathbb{R}$ can be identified with $\pi^{?}\zeta(p)\mathbb{Q}$, where ζ is the Riemann ζ-function.

1.3. **Goals of this lecture series.**

(a) We want to make precise the idea that Deligne cohomology is the analog of motivic cohomology for complex manifolds.

(b) A second goal is to understand the Hodge conjecture: For a complex smooth projective manifold M we have:

$$\begin{tikzcd} K_0(\mathrm{Vect}(M))_{\mathbb{Q}} \arrow[r, "\mathrm{ch}"] \arrow[dr, two heads] & \bigoplus_p \mathrm{H}^{2p}(M; \mathbb{Q}(p)) \arrow[d, two heads] \\ & \bigoplus_p \mathrm{Hdg}^p(M)_{\mathbb{Q}}. \end{tikzcd}$$

We want to modify $K_0(\mathrm{Vect}(M))$ using continuous K-theory to get a theory where it is reasonable to conjecture that $K_0(\mathrm{Nuc}(M))_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus_p \mathrm{H}^{2p}(M; \mathbb{Q}(p))$.

(c) A third goal is to make Riemann–Roch more transparent.

2. Talk 2: Complex manifolds from a new perspective (joint with P. Scholze)

Let M be a complex manifold. The Deligne cohomology of M was defined as a complex $\mathbb{Z}(p)_{\mathrm{D}}$ of sheaves, for any $p \in \mathbb{Z}$, given by the pullback

$$\begin{tikzcd} \mathbb{Z}(p)_{\mathrm{D}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (2\pi i)^p \mathbb{Z} \arrow[d] \\ F^p \Omega_{\mathrm{dR}} \arrow[r] & \Omega_{\mathrm{dR}}. \end{tikzcd}$$

Main goal: "Fix" the fact that

$$K_0(\mathrm{Vect}(M)) \to \bigoplus_p \mathrm{H}^{2p}(M; \mathbb{Z}(p)_{\mathrm{D}})$$

is far from being an isomorphism by replacing $K(\mathrm{Vect}(M))$ with $K^{\mathrm{cont}}(\mathrm{Nuc}(M))$. Here, $\mathrm{Nuc}(M)$ is some version of $D_{\mathrm{qc}}(M)$.

Idea: make it more like scheme theory: Start with a class of rings, R, which will determine everything: there is

- an underlying topological space,
- a structure sheaf of holomorphic functions,
- a de Rham complex,
- ....

The basic example of an R will be

$$\mathcal{O}^{\mathrm{hol}}(\mathbb{D}) = \left\{ \sum_n c_n T^n \,\middle|\, \exists r > 1 \text{ such that } \sum_n |c_n| r^n \to \infty \right\},$$

where $\mathbb{D} \subseteq \mathbb{C}$ is the *closed* unit disk and $\mathcal{O}^{\mathrm{hol}}$ denotes the functions which are holomorphic in a neighborhood of $\mathbb{D}$.

Good news! The abstract algebra $\mathcal{O}^{\mathrm{hol}}(\mathbb{D})$ determines $\mathbb{D}$. More precisely, there are mutually inverse maps

$$\begin{tikzcd} \mathbb{D} \arrow[rr, bend right, "x \mapsto \mathrm{ev}_x"'] & & \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}^{\mathrm{hol}}(\mathbb{D}), \mathbb{C}) \arrow[ll, bend right, "\varphi(T) \leftarrow\!\shortmid \varphi"'] \end{tikzcd}$$

Proof. We need:

(i) $\varphi(T) \in \mathbb{D}$. Suppose $\lambda \in \mathbb{C} \smallsetminus \mathbb{D}$. Then $\frac{1}{T-\lambda} \in \mathcal{O}^{\mathrm{hol}}(\mathbb{D})$, that is, $T - \lambda$ is a unit, hence so is $\varphi(T-\lambda) = \varphi(T) - \varphi(\lambda) \neq 0$, which is a contradiction.

(ii) φ is determined by $\varphi(T)$. This follows from the claim that

$$\lim_{N \to \infty} \varphi\Big(\sum_{n \leq N} c_n T^n\Big) = \varphi\Big(\sum_n c_n T^n\Big).$$

The same argument as above shows that $\varphi(\sum_{n>N} c_n T^n) \in \varepsilon\mathbb{D}$ for N such that $\sum_{n>N} |c_n| < \varepsilon$. $\square$

Definition 2.1. Let R be a $\mathbb{C}$-algebra. Define

$$\mathcal{M}_B(R) = \mathrm{Hom}_{\mathbb{C}}(R, \mathbb{C}) \subseteq \prod_{f \in R} \mathbb{C}$$

with the product topology.[1]

Claim. Both maps above are continuous.

The proof of the claim is (more or less) obvious. We now have that

$$\mathbb{D} = \mathcal{M}_B(\mathcal{O}^{\mathrm{hol}}(\mathbb{D})).$$

Bad news: We cannot get a structure sheaf, de Rham cohomology etc. just from the abstract $\mathbb{C}$-algebra structure on $\mathcal{O}^{\mathrm{hol}}(\mathbb{D})$. The most basic reason is that

$$\mathcal{O}^{\mathrm{hol}}(\mathbb{D}) \otimes_{\mathbb{C}} \mathcal{O}^{\mathrm{hol}}(\mathbb{D}) \neq \mathcal{O}^{\mathrm{hol}}(\mathbb{D}^2),$$

where $\otimes_{\mathbb{C}}$ is the abstract tensor product.

Solution: remember the topological vector space structure on $\mathcal{O}^{\mathrm{hol}}(\mathbb{D})$ and use the *completed* tensor product $\otimes_{\mathbb{C}}$. Or rather, use a category-friendly version thereof.

Concretely, this means that "topological" vector spaces are replaced with "light condensed" $\mathbb{C}$-vector spaces; then completeness corresponds to "gaseous".

Definition 2.2. A *light condensed abelian group* is a presheaf of abelian groups on $\mathrm{Pro(Fin)}^{\mathrm{light}}$, the category of countable inverse limits of finite sets, satisfying descent with respect to (1) finite coproducts and (2) surjections $S \twoheadrightarrow T$.

Exercise: We have that $\mathbb{N} \cup \{\infty\}$ lies in $\mathrm{Pro(Fin)}^{\mathrm{light}}$.

Example 2.3. For $\mathbb{C}$, there is a light condensed ring given by $S \mapsto \mathcal{C}^0(S, \mathbb{C})$, where $\mathcal{C}^0(S, \mathbb{C})$ denotes set of continuous functions from S to $\mathbb{C}$.

We can now consider the symmetric monoidal category

$$\big(\mathrm{Mod}_{\mathbb{C}}(\mathrm{CondAb}^{\mathrm{light}}), \otimes_{\mathbb{C}}\big).$$

We need to pass to a full subcategory of "complete" objects.

Idea: Completeness of M corresponds to the following property: if $m_0, m_1, m_2, \dots$ is a null sequence in M, then we can form $\sum_n m_n \cdot (1/2)^n \in M$.

Here, a *null sequence* is a map from $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}\infty$, the free condensed abelian group on a null sequence.

Definition 2.4. A module $M \in \mathrm{Mod}_{\mathbb{C}}(\mathrm{CondAb}^{\mathrm{light}})$ is called *gaseous* if the map

$$\mathrm{Null}(M) := \underline{\mathrm{Hom}}(\mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}\infty, M) \xrightarrow[\simeq]{1 - T \cdot \frac{1}{2}} \underline{\mathrm{Hom}}(\mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}\infty, M),$$

where T is induced by the shift map $\mathbb{N} \to \mathbb{N}$, $n \mapsto n+1$.

Remark 2.5. The condition for $\frac{1}{2}$ is equivalent to the condition for any λ with $0 < |\lambda| < 1$.

Theorem 2.6. *The full subcategory* $\mathrm{Mod}_{\mathbb{C}^{\mathrm{gas}}} \subseteq \mathrm{Mod}_{\mathbb{C}}(\mathrm{CondAb}^{\mathrm{light}})$ *of gaseous* $\mathbb{C}$*-vector spaces is abelian, closed under all colimits, limits, extensions, all* $\mathrm{R}^i \varprojlim$, $\mathrm{L}^i \varinjlim$ *and* $\mathrm{R}^i \underline{\mathrm{Hom}}(X, -)$*, for all* $X \in \mathrm{Mod}_{\mathbb{C}}(\mathrm{CondAb}^{\mathrm{light}})$.

[1] The subscript "B" stands for Betti or Berkovich.

Proof. Everything follows from the (interesting) fact that $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}\infty$ is *(internally) projective* in $\mathrm{Mod}_{\mathbb{C}}(\mathrm{CondAb}^{\mathrm{light}})$. ☐

Upshot:

(i) There exists a left adjoint

$$(-)^{\mathrm{gas}} \colon \mathrm{Mod}_{\mathbb{C}}(\mathrm{CondAb}^{\mathrm{light}}) \to \mathrm{Mod}_{\mathbb{C}^{\mathrm{gas}}}$$

to the inclusion,

(ii) There exists a symmetric monoidal structure on $\mathrm{Mod}_{\mathbb{C}^{\mathrm{gas}}}$ making $(-)^{\mathrm{gas}}$ symmetric monoidal.

(iii) There is a derived analog of everything.

Example 2.7. Any Banach space over $\mathbb{C}$ is gaseous. In particular, $\mathcal{O}^{\mathrm{hol}}(\mathbb{D})$ is gaseous (it is a filtered union of ℓ^1-spaces).

Theorem 2.8. *The ring $\mathcal{O}^{\mathrm{hol}}(\mathbb{D})$ is flat with respect to $- \otimes_{\mathbb{C}^{\mathrm{gas}}} -$ and*

$$\mathcal{O}^{\mathrm{hol}}(\mathbb{D}) \otimes_{\mathbb{C}^{\mathrm{gas}}} M = \varinjlim \mathrm{Null}(M).$$

Proof. Use trace class map tricks. ☐

The rings R that we consider are objects of $\mathrm{CAlg}(D_{\geq 0}(\mathbb{C}^{\mathrm{gas}}))$, which we call *gaseous $\mathbb{C}$-algebras*.

Definition 2.9. Let R be a gaseous $\mathbb{C}$-algebra.

(a) An element $f \in \pi_0 R(*)$ is called *topologically nilpotent* if there is a factorization

$$\begin{tikzcd} \mathbb{C}[T] \arrow[r, "T \mapsto f"] \arrow[d] & R \\ \mathbb{C}[\mathbb{N} \cup \{\infty\}/\infty] \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

of condensed rings.

(b) R is called *pointwise bounded* if for all $f \in R$ (meaning: $f \in \pi_0 R(*)$), there exists $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$ such that λf is topologically nilpotent.

Definition 2.10 (Rodriguez-Camargo)**.** Let $S \in \mathrm{Pro}(\mathrm{Fin})^{\mathrm{light}}$. Then $f \in R(S)$ is *uniformly topologically nilpotent* if there exists a factorization

$$\begin{tikzcd} \mathbb{C}\left[\bigsqcup_d \mathrm{Sym}^d S\right] \arrow[r, "f"] \arrow[d] & R \\ \mathbb{C}\left[\left(\bigsqcup_d \mathrm{Sym}^d S\right) \cup \{\infty\}/\infty\right]. \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

R is called *bounded* if for all S and all $f \in R(S)$, there exists $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$ such that λf is uniformly topologically nilpotent.

Theorem 2.11 (Rodriguez-Camargo)**.** $\mathrm{CAlg}(D_{\geq 0}(\mathbb{C}^{\mathrm{gas}}))^{\mathrm{bded}} \subseteq \mathrm{CAlg}(D_{\geq 0}(\mathbb{C}^{\mathrm{gas}}))$ *is closed under all colimits and finite limits.*

Example 2.12. Any Banach algebra R over $\mathbb{C}$ is bounded. In particular, $\mathcal{O}^{\mathrm{hol}}(\mathbb{D})$ is bounded.

Theorem 2.13. *If R is pointwise bounded, then $\mathcal{M}_B(R(*))$ is compact Hausdorff and*

$$D(R) \coloneqq \mathrm{Mod}_R(D(\mathbb{C}^{\mathrm{gas}}))$$

localizes along $\mathcal{M}_B(R(*))$ (the abstract $\mathbb{C}$-algebra underlying $\mathcal{M}_B(R)$).

3. Talk 3: Quasicoherent sheaves in complex geometry (joint with P. Scholze)

Recall, we considered $R \in \mathrm{CAlg}(D_{\geq 0}(\mathbb{C}^{\mathrm{gas}}))^{\mathrm{bded}}$, meaning that for all $f \in R$ there exists $\lambda \in \mathbb{C}^{\times}$ such that λf is topologically nilpotent and similarly for $f \in R(S)$ with $S \in \mathrm{Pro}(\mathrm{Fin})^{\mathrm{light}}$.[2]

`Clausen:Gelfand`

Theorem 3.1. *The category $D(R) := \mathrm{Mod}_R(D(\mathbb{C}^{\mathrm{gas}}))$ localizes on the compact Hausdorff space $\mathcal{M}_B(R(*)) = \mathrm{Hom}_{\mathbb{C}}(R(*), \mathbb{C})$, called the* Gelfand spectrum*.*

Note that under the boundedness condition we even have a closed embedding

$$\mathcal{M}_B(R(*)) \subseteq \prod_{f \in R(*)} \mathbb{C}_{|\cdot| \leq C_f},$$

where $C_f \in \mathbb{R}_{>0}$ depends on f.

Example 3.2. If $R = \mathcal{O}^{\mathrm{hol}}(\mathbb{D}^n)$, then $\mathcal{M}_B(R(*)) = \mathbb{D}^n$.

Definition 3.3 (Balmer–Krause–Stevenson)**.** Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L)$. Define a full subcategory

$$\mathrm{Idem}(\mathcal{C}) \subseteq \mathrm{CAlg}(\mathcal{C})$$

consisting of the *idempotent* algebras, i.e., algebras R such that $\mathbf{1} \to R$ induces an isomorphism $R = \mathbf{1} \otimes R \xrightarrow{\sim} R \otimes R$ (equivalently, $m\colon R \otimes R \xrightarrow{\sim} R$ is an isomorphism).

Proposition 3.4. *The category* $\mathrm{Idem}(\mathcal{C})$ *is a poset.*[3]

Moreover, $\mathrm{Idem}(\mathcal{C})$ *has arbitrary colimits and finite limits, which are calculated as follows:*

(1) sifted colimits are calculated in $\mathcal{C}$.
(2) finite coproducts are calculated by $\otimes$.
(3) pullbacks are computed as the fiber product

$$\begin{tikzcd} \text{(pullback)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d] \\ S \arrow[r] & S \otimes R. \end{tikzcd}$$

Moreover, $\mathrm{Idem}(\mathcal{C})$ *is a* locale *(i.e., it satisfies the same properties as open subsets of a topological space).*

A more precise version of Theorem 3.1 is the following:

Theorem 3.5. *There exists a map of posets (locales$^{\mathrm{op}}$?)*

$$\mathrm{Closed}(\mathcal{M}_B(R)) \to \mathrm{Idem}(D(R))^{\mathrm{op}}$$

preserving finite colimits and limits.

The map is uniquely determined by the following: For all $f \in R$ and $C \in \mathbb{R}_{>0}$ it is given by

$$\{|f| \leq C\} \mapsto R \otimes_{\mathbb{C}^{\mathrm{gas}}} \mathcal{O}(C \cdot \mathbb{D})/(T - f),$$
$$\{|f| \geq C\} \mapsto R \otimes_{\mathbb{C}^{\mathrm{gas}}} \mathcal{O}\big(\{|M| \geq C\} \text{ merom. at } \infty\big)/(T - f).$$

Explicitly, for any closed subset $K \subset \mathcal{M}_B(R)$ we get an idempotent R-algebra $\mathcal{O}(K)$.

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[2] An analogous definition was considered by Ralf Meyer.
[3] This means that the anima of maps is either empty or contractible.

Example 3.6. Let $R = \mathcal{O}(\mathbb{D}^n)$. Then

$$\mathcal{O}(K) = \varinjlim_{\substack{U \supseteq K \\ \text{open}}} \mathrm{R}\Gamma(U; \mathcal{O}^{\mathrm{hol}}).$$

Remark 3.7. In general, $\mathcal{O}(K)$ can live in positive and negative degrees. In practice it lives in degrees ≤ 0.

Moreover, in general, $\{K \mid \mathcal{O}(K) \in D_{\geq 0}(\mathbb{C}^{\mathrm{gas}})\}$ is closed under intersections and generates the topology. For any such K, we have $\mathcal{O}(K) \in \mathrm{CAlg}(D_{\geq 0}(\mathbb{C}^{\mathrm{gas}}))^{\mathrm{bded}}$ and

$$\begin{tikzcd} \mathcal{M}_B(\mathcal{O}(K)) \arrow[r] \arrow[d, leftrightarrow, "\simeq"'] & \mathcal{M}_B(R) \\ K, \arrow[ur, hook] & \end{tikzcd}$$

and for $K' \subseteq K$ the idempotent algebras agree. (This is analogous to distinguished opens in algebraic geometry.)

Example 3.8. If $R = \mathcal{O}(\mathbb{D}^n)$, then $K \subseteq \mathbb{D}^n$ satisfies $\mathcal{O}(K) \in D_{\geq 0}(\mathbb{C}^{\mathrm{gas}})$ if and only if K is holomorphic (?) convex (compact Stein).

Recall (Lurie), if X is locally compact Hausdorff and $\mathcal{C} \in \mathrm{Pr}^L_{\mathrm{st}}$, then

$$\mathrm{Shv}(X; \mathcal{C}) \xleftrightarrow{\simeq} \mathrm{Shv}_K(X; \mathcal{C}),$$

where the right hand side is the category of presheaves on compact subsets such that (1) it satisfies the sheaf condition for finite covers and (2) $\mathcal{F}(K) = \varinjlim_{K \Subset K'} \mathcal{F}(K')$.

For $X = \mathcal{M}_B(R)$ one has the same if one only restricts to K such that $\mathcal{O}(K) \in D_{\geq 0}(\mathbb{C}^{\mathrm{gas}})$.

Corollary 3.9. *(a) We get a structure sheaf* $\mathcal{O} \in \mathrm{Shv}(\mathcal{M}_B(R); \mathrm{CAlg}(D(\mathbb{C}^{\mathrm{gas}})))$. *(For* $R = \mathcal{O}(\mathbb{D})$ *we get the usual* $\mathcal{O}^{\mathrm{hol}}$*.)*
(b) We get a sheaf with values in $\mathrm{CAlg}(\mathrm{Pr}^L)$*, given by* $K \mapsto D(\mathcal{O}(K))$*; this uses idempotency.*

Theorem 3.10 (Automatic quasicoherence)**.** *The functor*

$$D(R) \to \mathrm{Mod}_{\mathcal{O}}(\mathrm{Shv}(\mathcal{M}_B(R); D(\mathbb{C}^{\mathrm{gas}}))),$$
$$M \mapsto \big(K \mapsto M \otimes_R \mathcal{O}(K)\big)$$

is an equivalence.

Proof. Fully faithfulness is easy. For essential surjectivity it is enough to hit the generators "h_U", where $U \subseteq \mathcal{M}_B(R)$ is open. These are hit by $\mathrm{fib}(R \to \mathcal{O}(X \smallsetminus U))$. $\square$

We thus get a category $D_{\mathrm{qcoh}}(M)$ for any complex manifold M.

Theorem 3.11. *Let R be bounded. Then we can define a (derived de Rham) complex on* $\mathcal{M}_B(R)$*, which is also a sheaf.*

Example 3.12. If $R = \mathcal{O}(\mathbb{D}^n)$, then we get back the usual de Rham complex.
The key input is that $\mathbb{C}[T_1, \ldots, T_n] \hookrightarrow \mathcal{O}(\mathbb{D}^n)$ is idempotent.

Note that this also allows us to define Deligne cohomology.

Warning 3.13. The category $D(\mathbb{C}^{\mathrm{gas}})$ is not rigid.

The fix is to pass to a full subcategory which *is* rigid. Recall that, if $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ is such that $\mathbf{1}$ is compact, then $\mathcal{C}$ is rigid if and only if $\mathcal{C}$ is generated by *basic nuclear* objects (which is the same as ω_1-compact objects), i.e., objects of the form $\varinjlim(x_0 \to x_1 \to \cdots)$, where all transition maps $x_n \to x_{n+1}$ are trace class.[4] This is not satisfied for $\mathcal{C} = D(\mathbb{C}^{\mathrm{gas}})$. For example, $P = \mathbb{C}^{\mathrm{gas}}(\mathbb{N} \cup \{\infty\}/\infty)$ is compact in $D(\mathbb{C}^{\mathrm{gas}})$. But it is not basic nuclear, because $\mathrm{id}\colon P \to P$ is not trace class.

In fact, we have $P \subseteq \prod_{\mathbb{N}} \mathbb{C}$ consisting of those sequences with "quasi-exponential decay". The trace class maps in $D(\mathbb{C}^{\mathrm{gas}})$ are generated by maps $P \to P$ which are given by a diagonal matrix with quasi-exponential decay.

Definition 3.14. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ and assume that $\mathbf{1}$ is compact. Let

$$\mathrm{Nuc}(\mathcal{C}) \subseteq \mathcal{C}$$

be the full subcategory generated by the basic nuclear objects.

Fact 3.15. $\mathrm{Nuc}(\mathcal{C})$ is closed under $\otimes$, and we have $\mathbf{1} \in \mathrm{Nuc}(\mathcal{C})$.

Question 3.16. Is $\mathrm{Nuc}(\mathcal{C})$ always rigid?

In general, the answer is *no*! But the answer is yes if every trace class map factors as the composite of two trace class maps. This holds for $D(\mathbb{C}^{\mathrm{gas}})$, i.e., $\mathrm{Nuc}(\mathbb{C}^{\mathrm{gas}})$ is rigid.

Theorem 3.17. *(a)* $\mathrm{Nuc}(\mathbb{C}^{\mathrm{gas}}) = \mathrm{Nuc}(\langle P \rangle)$.

(b) Let $R \in \mathrm{CAlg}(D(\mathbb{C}^{\mathrm{gas}}))$ *such that* $R \in \mathrm{Nuc}(\mathbb{C}^{\mathrm{gas}})$*, then*

$$\mathrm{Nuc}(\mathrm{Mod}_R(D(\mathbb{C}^{\mathrm{gas}}))) = \mathrm{Mod}_R(\mathrm{Nuc}(\mathbb{C}^{\mathrm{gas}})).$$

(c) $\mathcal{O}^{\mathrm{hol}}(\mathbb{D}^n) \in \mathrm{Nuc}(\mathbb{C}^{\mathrm{gas}})$.

(d) $\mathrm{Nuc}(\mathbb{C}^{\mathrm{gas}}) \subseteq D(\mathbb{C}^{\mathrm{gas}})$ *is closed under countable limits.*

Hence, for a complex manifold M, we can define

$$D_{\mathrm{qcoh}}(M) \supseteq \mathrm{Nuc}(M)$$

such that $\mathcal{F}$ is nuclear if and only if $\mathcal{F}(K)$ is a nuclear $\mathcal{O}(K)$-module for all compact K or, equivalently, $\mathcal{F}(K)$ is nuclear over $\mathbb{C}^{\mathrm{gas}}$.

4. Talk 4: Cohomology theories on complex manifolds (joint with P. Scholze)

Theorem 4.1. *If $R \neq 0$ is pointwise bounded, then there exists a $\mathbb{C}$-algebra morphism $R(*) \to \mathbb{C}$.*

Proof. If there were no such algebra morphism, then $R = \mathcal{O}(\mathcal{M}_B(R)) = \mathcal{O}(\varnothing) = 0$.

Challenge: Give a direct proof. □

Definition 4.2. We put

$$\begin{array}{ccc} \mathrm{Shv}(\mathrm{Man}_{\mathbb{C}}; \mathrm{Sp}) & = & \left\{ \mathcal{F}\colon \mathrm{Man}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Sp} \;\middle|\; \forall M \in \mathrm{Man}_{\mathbb{C}}, \mathcal{F}|_{\mathrm{Open}(M)} \text{ is a sheaf} \right\} \\ \downarrow{\scriptstyle \simeq} & & \\ \mathrm{Shv}(\mathcal{O}(C)\text{'s}; \mathrm{Sp}) & = & \left\{ \mathcal{F}\colon \{\mathcal{O}(C)\text{'s}\} \to \mathrm{Sp} \;\middle|\; \forall C, \mathcal{F}|_{\mathrm{Closed}(C)} \text{ is a } \mathcal{K}\text{-sheaf} \right\}, \end{array}$$

where the $\mathcal{O}(C)$'s, for $C \subseteq \mathbb{C}^d$ compact Stein, live in $\mathrm{CAlg}(D_{\geq 0}(\mathbb{C}^{\mathrm{gas}}))$.

[4] If $\mathcal{C}$ is compactly generated, then $X \in \mathcal{C}$ is nuclear if and only if $(\underline{\mathrm{Hom}}(K, \mathbf{1}) \otimes X)(*) \xrightarrow{\sim} \mathrm{Hom}(K, X)$ is an isomorphism for all compact K.

Example 4.3. (1) For all $p \in \mathbb{Z}$ we have $\mathbb{Z}(p)_{\mathrm{D}}(M) = \mathrm{R}\Gamma(M; \mathbb{Z}(p)_{\mathrm{D}})$.
(2) K^{nuc}: we have $K^{\mathrm{nuc}}(\mathcal{O}(C)) = K^{\mathrm{cont}}(\mathrm{Nuc}(\mathcal{O}(C)))$.
(3) K-theory analog (Karoubi)

$$\begin{tikzcd} K^{\mathrm{Del}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{HC}^-(\mathcal{O}(K)/\mathbb{C}^{\mathrm{gas}}) \arrow[d] \\ \underline{KU} \arrow[r] & \mathrm{HP}(\mathcal{O}(K)/\mathbb{C}^{\mathrm{gas}}), \end{tikzcd}$$

where the bottom map is induced by $ku \to \mathrm{HP}(\mathbb{C}/\mathbb{C}^{\mathrm{gas}}) = \prod_{n\in\mathbb{Z}} \mathbb{C}(2n)$.

There exists a filtration on K^{Del} with associated graded pieces $\mathbb{Z}(p)_{\mathrm{D}}[2p]$, which rationally splits:

$$K^{\mathrm{Del}}_{\mathbb{Q}} = \bigoplus_p \mathbb{Q}(p)_{\mathrm{D}}[2p].$$

Theorem 4.4. *There exists a natural map $K^{\mathrm{nuc}} \to K^{\mathrm{Del}}$.*

Main Conjecture 4.5 (Modified Hodge Conjecture)**.** *The map above is an isomorphism.*

Theorem 4.6. *In any case, K^{Del} (and all three terms in its definition) is an invariant of $\mathrm{Nuc}(\mathcal{O}(K))$ (as a $\mathrm{Nuc}(\mathbb{C}^{\mathrm{gas}})$-linear category).*

We obtain proper pushforward, which rather easily implies Riemann–Roch theorems.

Definition 4.7. A sheaf $\mathcal{F} \in \mathrm{Shv}(\mathrm{Man}_{\mathbb{C}}; \mathrm{Sp})$ is $\mathbb{D}^0$-*invariant* if the pullback map

$$\mathcal{F}(M) \xrightarrow{\sim} \mathcal{F}(\mathbb{D}^0 \times M)$$

is an isomorphism for all $M \in \mathrm{Man}_{\mathbb{C}}$.

Theorem 4.8. *(a) For $\mathcal{F} \in \mathrm{Shv}(\mathrm{Man}_{\mathbb{C}}; \mathrm{Sp})$ the following are equivalent:*
(1) $\mathcal{F}$ is $\mathbb{D}^0$-invariant.
(2) $\mathcal{F}$ (viewed as a $\mathcal{K}$-sheaf) is $[0,1]$-invariant (where $[0,1] \subset \mathbb{C}$).
(3) For all $d \geq 0$, the map $\mathcal{F}(\mathcal{O}_{\mathbb{C}^d,0}) \xrightarrow{\sim} \mathcal{F}(\mathbb{C})$ is an isomorphism.
(b) The functor $\mathrm{Shv}_{\mathbb{D}^0}(\mathrm{Man}_{\mathbb{C}}; \mathrm{Sp}) \xrightarrow{\sim} \mathrm{Sp}$, $\mathcal{F} \mapsto \mathcal{F}(*)$ is an equivalence.
(c) The inclusion $\mathrm{Shv}_{\mathbb{D}^0}(\mathrm{Man}_{\mathbb{C}}; \mathrm{Sp}) \subseteq \mathrm{Shv}(\mathrm{Man}_{\mathbb{C}}; \mathrm{Sp})$ has a left and a right adjoint.
The right adjoint is given by $\mathcal{F} \mapsto (M \mapsto \mathrm{R}\Gamma(M; \mathcal{F}(*)))$.
The left adjoint is given by $\mathcal{F} \mapsto \mathcal{F}^h := \varinjlim_{[n]\in\Delta} \mathcal{F}(- \times \Delta^n)$, where $\mathcal{F}$ is viewed as a $\mathcal{K}$-sheaf.

Example 4.9. (i) Let $M \in \mathrm{Man}_{\mathbb{C}}$. Then

$$\mathbb{S}[h_M]^h = \mathrm{R}\Gamma(-; \mathbb{S}(\Pi_\infty M)).$$

(ii) The "homotopification" of the sheafification $\big(K(\widetilde{\mathrm{Vect}(-)})\big)^h$ identifies with $\mathrm{R}\Gamma(-, ku)$.
`Clausen:thm`

Theorem 4.10. *(a) We have*

$$(K^{\mathrm{nuc}})^h = \mathrm{R}\Gamma(-; KU).$$

(b) The map $K^{\mathrm{nuc}} \to (K^{\mathrm{nuc}})^h$ is an isomorphism in degrees ≤ 0 (on any C).
(c) $(\mathrm{HC}^-)^h = \mathrm{HP}$.

The theorem implies the existence of a natural trace map

$$\begin{tikzcd} K^{\mathrm{nuc}} \arrow[dr] \arrow[drr, bend left, "\exists\ \mathrm{trace}"] \arrow[ddr, bend right] & & \\ & K^{\mathrm{Del}} \arrow[r] \arrow[d] & \mathrm{HC}^- \arrow[d] \\ & (K^{\mathrm{nuc}})^h \arrow[r] & (\mathrm{HC}^-)^h, \end{tikzcd}$$

which uses rigidity of $\mathrm{Nuc}(\mathbb{C}^{\mathrm{gas}})$.

Corollary 4.11. *The following are equivalent:*

(a) The modified Hodge conjecture holds.
(b) $\mathrm{fib}(K^{\mathrm{nuc}} \to \mathrm{HC}^-)$ *is a* $\mathbb{D}^0$*-invariant sheaf.*
(c) $\dagger$*-rigidity: the commutative square*

$$\begin{tikzcd} K^{\mathrm{nuc}}(\mathcal{O}_{\mathbb{C}^\dagger,0}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{TC}^-(\mathcal{O}_{\mathbb{C}^d,0}/\mathbb{C}^{\mathrm{gas}}) \arrow[d] \\ K^{\mathrm{nuc}}(\mathbb{C}) \arrow[r] & \mathrm{TC}^-(\mathbb{C}/\mathbb{C}) \end{tikzcd}$$

is a pullback.

Remark 4.12. Replace $\mathbb{C}^{\mathrm{gas}}$ by $\mathbb{Q}_p^{\mathrm{solid}}$. Then $\dagger$-rigidity is true!

Remark 4.13 (Conjecture)**.** $\mathrm{HH}(\mathcal{O}(C)/\mathbb{C}^{\mathrm{gas}}) = \mathrm{HH}^{\mathrm{cont}}(\mathrm{Nuc}(\mathcal{O}(C))/\mathbb{Q})$.
Again, this is true in the non-archimedean analog (due to Cordova).

Proof of Theorem 4.10. **Step 1:** For (a), use the commutative diagram

$$\begin{tikzcd} ku \arrow[r, equal] & (K^{\mathrm{vect}})^h \arrow[r] \arrow[d] & (K^{\mathrm{nuc}})^h \\ & KU, \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

i.e., $\beta \in \pi_2(ku)$ is invertible in $(K^{\mathrm{nuc}})^h$.
For the proof, use GAGA: If X is a smooth proper scheme over $\mathbb{C}$, then

$$D_{\mathrm{qc}}(X) \otimes_{D(\mathbb{C}(*))} \mathrm{Nuc}(\mathbb{C}^{\mathrm{gas}}) \xrightarrow{\sim} \mathrm{Nuc}_{\mathrm{qc}}(X^{\mathrm{an}}).$$

Apply this to $X = \mathbb{P}^1$. Then $\mathrm{Nuc}(\mathbb{P}^1) = \langle \mathrm{Nuc}(\mathbb{C}^{\mathrm{gas}}), \mathrm{Nuc}(\mathbb{C}^{\mathrm{gas}}) \rangle$, which implies

$$K^{\mathrm{nuc}}(\mathbb{P}^1 \times C) \simeq K^{\mathrm{nuc}}(C) \oplus K^{\mathrm{nuc}}(C).$$

Applying $(-)^h$, we deduce an inverse for β.

Step 2: The sheaf $\pi_0 K^{\mathrm{nuc}}(\mathcal{O}(-))$ is $[0,1]$-invariant.

Proof. The map $K(\mathcal{O}(C)(*)) \to K^{\mathrm{nuc}}(C) = K^{\mathrm{cont}}(\mathrm{Nuc}(\mathcal{O}(C)))$ is an isomorphism on π_0 (and only on π_0!). "The proof is tricky but sort of straightforward."
By Grauert–Oka, $K_0(\mathrm{Vect}(\mathcal{O}(C)))$ is homotopy invariant. ☐

Step 3: The sheaf $\tau_{\leq 0}K^{\mathrm{nuc}}$ is $[0,1]$-invariant.

For the proof, one uses Bass delooping to compute

$$K^{\mathrm{nuc}}_{-1}(C) = \mathrm{coker}\big(K^{\mathrm{nuc}}_0(\mathbb{D}_+ \times C) \oplus K^{\mathrm{nuc}}_0(\mathbb{D}_- \times C) \to K^{\mathrm{nuc}}_0(S^1 \times C)\big)$$

and then do descending induction.

Step 4: The map $K^{\mathrm{nuc}} \to (K^{\mathrm{nuc}})^h$ is an isomorphism on $\tau_{\leq 0}$.

We have $(K^{\mathrm{nuc}})^h = \varinjlim K^{\mathrm{nuc}}(- \times \Delta^n)$, so the claim follows immediately from Step 3.

Step 5: We have $(K^{\mathrm{nuc}})^h = \mathrm{R}\Gamma(-;(K^{\mathrm{nuc}})^h(*))$, which is a KU-module by Step 1, hence is 2-periodic. But on the other hand, the map $K^{\mathrm{nuc}} \to (K^{\mathrm{nuc}})^h$ is an isomorphism on π_0 and π_{-1}.

It thus suffices to show

$$\pi_0 K^{\mathrm{nuc}}(\mathbb{C}) = \mathbb{Z},$$
$$\pi_{-1} K^{\mathrm{nuc}}(\mathbb{C}) = 0.$$

$\square$