CONTINUOUS K-THEORY AND GEOMETRIC TOPOLOGY

THOMAS NIKOLAUS

Contents

1. Talk 1

Recall that $\mathrm{Pr}^L_{\mathrm{st}}$ is the category of presentable stable categories. We denote by $\otimes$ the Lurie tensor product; then $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ is the internal Hom and the category Sp of spectra is the unit for $\otimes$.

Definition 1.1. $\mathcal{C}$ is *dualizable* if $\mathcal{C}$ is dualizable in $(\mathrm{Pr}^L_{\mathrm{st}}, \otimes)$. The dual is then given by $\mathcal{C}^\vee = \mathrm{Fun}^L(\mathcal{C}, \mathrm{Sp})$.

Definition 1.2. (i) A functor $F\colon \mathcal{D} \to \mathcal{E}$ is called a *homological epi* if the restriction $\mathrm{Ind}(\mathcal{E}) \to \mathrm{Ind}(\mathcal{D})$ is fully faithful or, equivalently, if the induced functor $\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$ is a Bousfield localization.

(ii) A map $R \to S$ of ring spectra is a *homological epi* if the following equivalent conditions hold:
- the induced functor $\mathrm{Mod}(R)^\omega \to \mathrm{Mod}(S)^\omega$ is a homological epi.
- $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$ is a Bousfield localization.
- the restriction $\mathrm{Mod}(S) \to \mathrm{Mod}(R)$ is fully faithful.
- $S \otimes_R S \xrightarrow{\sim} S$.
- $S \sqcup_R S \xrightarrow{\sim} S$, where the pushout is taken in $\mathbb{E}_1$-rings.
- $I \otimes_R I \xrightarrow{\sim} I$, where $I = \mathrm{fib}(R \to S)$.

(iii) A non-unital ring spectrum R is called *H-unital* if $R^+ = R \oplus \mathbb{S} \to \mathbb{S}$ is a homological epi, where $\mathbb{S}$ is the sphere spectrum.

Definition 1.3. (i) A map $x \to y$ in $\mathcal{C}$ is called *(weakly) compact* if for every filtered colimit $d = \mathrm{colim}_i\, d_i$ and every map $y \to d$, the composite $x \to y \to d$ factors over some d_{i_0}:

$$\begin{array}{ccc} x & \longrightarrow & y \\ {\scriptstyle \exists}\,\downarrow & & \downarrow \\ d_{i_0} & \longrightarrow & \mathrm{colim}_i\, d_i. \end{array}$$

(ii) An object $x \in \mathcal{C}$ is called *compactly exhaustible* if it can be written as

$$x = \operatorname{colim}(x_0 \to x_1 \to \cdots)$$

with compact transition maps $x_i \to x_{i+1}$.

(iii) An object $x \in \mathcal{C}$ is called *transfinitely compactly exhaustible* if it can be written as

$$x = \operatorname*{colim}_{i \in I} x_i,$$

where I is filtered, without terminal object, and *antisymmetric* (i.e., the non-invertible morphisms in I form an ideal; equivalently one-sided inverses are invertible), such that for every non-invertible map $i \to j$ in I, the induced map $x_i \to x_j$ is compact.

Nikolaus:dualizable

Theorem 1.4. *For $\mathcal{C} \in \mathrm{Pr}^L_{\mathrm{st}}$ the following are equivalent:*

(1) $\mathcal{C}$ is dualizable.

(2) $\mathcal{C}$ is a retract in $\mathrm{Pr}^L_{\mathrm{st}}$ of a compactly generated stable category, i.e., $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$ for a small stable category $\mathcal{C}_0$.

(3) $\mathcal{C}$ is the kernel of

$$\mathrm{Ind}(\mathcal{D}) \xrightarrow{\mathrm{Ind}(F)} \mathrm{Ind}(\mathcal{E}),$$

where $\mathcal{D}, \mathcal{E}$ are small stable categories and $F\colon \mathcal{D} \to \mathcal{E}$ is an exact homological epi.

(4) $\mathcal{C}$ is the kernel of

$$\mathrm{Mod}(R) \to \mathrm{Mod}(S)$$

for a map $R \to S$ of ring spectra which is a homological epi. In this case we write $\mathcal{C} = \mathrm{Mod}(R, I)$.

(5) $\mathcal{C}$ is the kernel of

$$\mathrm{Mod}(R^+) \to \mathrm{Mod}(\mathbb{S}) = \mathrm{Sp},$$

where R is an H-unital ring spectrum. In this case we write $\mathcal{C} = \mathrm{Mod}_{\mathrm{H}}(R)$.

(6) The colimit functor $k\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\hat{\jmath}$ (which is automatically fully faithful since the right adjoint j of k is fully faithful).

(7) $\mathcal{C}$ is ω_1-compactly generated and the colimit

$$k\colon \mathrm{Ind}(\mathcal{C}^{\omega}_1) \to \mathcal{C}$$

admits a left adjoint.

(8) $\mathcal{C}$ is generated under colimits by compactly exhaustible objects.

(9) Every object in $\mathcal{C}$ is transfinitely compactly exhaustible.

(10) (AB6) Products in $\mathcal{C}$ distribute over filtered colimits: the natural map

$$\operatorname*{colim}_{(i_k)_k \in \prod_{k \in K} I_k} \prod_{k \in K} x_{k,i_k} \xrightarrow{\sim} \prod_{k \in K} \operatorname*{colim}_{i \in I_k} x_{k,i}$$

is an isomorphism in $\mathcal{C}$.[1]

[1]Since we are in a stable category, we may equivalently replace products with limits. Moreover, one can assume that all I_k's are the same.

"Proofs": If $\mathcal{C}$ is dualizable, we have a Bousfield localization $\mathrm{Ind}(\mathcal{C}^\kappa) \to \mathcal{C}$. We then obtain a lifting diagram

$$\begin{array}{ccc} \mathrm{Ind}(\mathcal{C}^\kappa)\otimes\mathcal{C}^\vee & \longrightarrow & \mathcal{C}\otimes\mathcal{C}^\vee \\ & \nwarrow & \uparrow{\scriptstyle \mathrm{coev}} \\ & & \mathrm{Sp}, \end{array}$$

where the lift exists, because a functor from Sp is uniquely determined by specifying the image of $\mathbb{S}$. Reinterpreting, we obtain a lift

$$\begin{array}{ccc} & & \mathrm{Ind}(\mathcal{C}^\kappa) \\ & \nearrow & \downarrow \\ \mathcal{C} & = & \mathcal{C}. \end{array}$$

For the implication (2) $\Longrightarrow$ (6), write $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$. Then we have

$$\hat{\jmath} = \mathrm{Ind}(\mathcal{C}_0 \hookrightarrow \mathcal{C})\colon \mathcal{C} = \mathrm{Ind}(\mathcal{C}_0) \to \mathrm{Ind}(\mathcal{C}).$$

To pass to retracts, use some abstract argument.

For the implication (6) $\Longrightarrow$ (2), use $\hat{\jmath}\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.

Ad (7) $\Longrightarrow$ (3): Consider

$$\mathcal{C} \xrightarrow{\hat{\jmath}} \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C} = \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})).$$

To see the equality, use that $\mathcal{C}^{\omega_1} \xrightarrow{\mathrm{pr}} \mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})$ is a homological epi.

Finally, for (6) $\iff$ (10), note that distributivity is equivalent to $k\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ preserving products. (This relies on the fact that the category of anima satisfies (AB6).) □

Example 1.5. The category $\mathrm{Shv}(X)$ of sheaves with values in Sp on a locally compact space X is generated by sheaves of the form $\Sigma^\infty_+ \underline{U}$, where $\underline{U}$ is the sheaf on X represented by U and $\Sigma^\infty_+\colon \mathrm{Ani} \to \mathrm{Sp}$ is the suspension functor. Every inclusion $U \hookrightarrow V$ in $\mathrm{Open}(X)$ factors through a compact subset K, that is, $U \to K \to V$. It follows that $\Sigma^\infty_+\underline{U} \to \Sigma^\infty_+\underline{V}$ is a compact map. Finally, $\Sigma^\infty_+\underline{U}$ is compactly exhaustible if U is, and every U is a filtered colimit of such. Therefore, $\mathrm{Shv}(X)$ is dualizable.

Properties 1.6 (of dualizable categories)**.** Let $\mathcal{C}, \mathcal{D}$ be dualizable categories.

(a) An object $x \in \mathcal{C}$ is compactly exhaustible if and only if x is ω_1-compact.

(b) A functor $F\colon \mathcal{C} \to \mathcal{D}$ in Pr^L is strongly left adjoint (meaning that F admits a right adjoint that preserves colimits) if and only if F preserves compact morphisms, if and only if the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Ind}(\mathcal{C}) & \xrightarrow{\mathrm{Ind}(F)} & \mathrm{Ind}(\mathcal{D}) \\ {\scriptstyle \hat{\jmath}}\uparrow & \Nwarrow^{\sim} & \uparrow{\scriptstyle \hat{\jmath}} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D}. \end{array}$$

(c) A morphism $x \to y$ in $\mathcal{C}$ is compact if it lifts to a map $jx \to \hat{\jmath}y$ in $\mathrm{Ind}(\mathcal{C})$. In this case, we define the space of *compactly assembled maps* as the spectrum

$$\mathrm{map}^{\mathrm{ca}}_{\mathcal{C}}(x, y) := \mathrm{map}_{\mathrm{Ind}(\mathcal{C})}(jx, \hat{\jmath}y).$$

(d) If $x = \mathrm{colim}_{i\in I}\, x_i$ is I-compactly exhaustible, then $\hat{\jmath}(x) = \mathrm{colim}_{i\in I}\, jx_i$ in $\mathrm{Ind}(\mathcal{C})$.

(e) Recall the resolution
$$\mathcal{C} \to \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})),$$
where $\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}) \coloneqq (\mathrm{Ind}(\mathcal{C}^{\omega_1})/\mathcal{C})^{\omega}$, and denote $p\colon \mathcal{C}^{\omega_1} \to \mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})$ the projection. Then
$$\mathrm{map}_{\mathrm{Calk}(\mathcal{C})}(px, py) = \mathrm{map}_{\mathcal{C}}(x, y)/\,\mathrm{map}^{\mathrm{ca}}_{\mathcal{C}}(x, y).$$

Definition 1.7. We denote $\mathrm{Pr}^L_{\mathrm{dual}}$ the category of dualizable categories with strongly left adjoint functors.

Corollary 1.8. *A left adjoint functor* $\mathrm{Shv}(X) \to \mathcal{D}$ *is strongly left adjoint if the corresponding cosheaf* $\mathcal{F}\colon \mathrm{Open}(X) \to \mathcal{D}$ *satisfies the following condition: for every* $U \Subset V$*, the induced map* $\mathcal{F}(U) \to \mathcal{F}(V)$ *is a compact morphism in* $\mathcal{D}$*.*

Definition 1.9. We define the continuous K-theory of a dualizable category $\mathcal{C}$ as the fiber
$$K^{\mathrm{cont}}(\mathcal{C}) = \mathrm{fib}\big(K(\mathcal{C}^{\omega_1}) \to K(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}))\big) \cong \Omega K(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})).$$

2. Talk 2: Verdier duality and 6 functors

2.1. **Recap.** Let $\mathcal{C}$ be a dualizable category. Recall that we have a resolution
$$\mathcal{C} \to \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathrm{Ind}\,\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}).$$
The category $\mathrm{Pr}^L_{\mathrm{dual}}$ of dualizable categories with strongly left adjoint functors is itself a presentable category (due to Ramzi), and the forgetful functor $\mathrm{Pr}^L_{\mathrm{dual}} \to \mathrm{Pr}^L$ preserves colimits.

Exercise 2.1. To compute colimits in Pr^L, use $\mathrm{Pr}^L \simeq (\mathrm{Pr}^R)^{\mathrm{op}}$ and use that $\mathrm{Pr}^R \to \mathrm{Cat}$ commutes with limits. Use this to prove that $\mathrm{Pr}^L_{\mathrm{dual}} \to \mathrm{Pr}^L$ commutes with colimits.

Definition 2.2. Let $\mathcal{C}$ be a dualizable category. We define the *continuous K-theory* as
$$K^{\mathrm{cont}}(\mathcal{C}) \coloneqq \mathrm{fib}\big(K(\mathcal{C}^{\omega_1}) \to K(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}))\big) \simeq \Omega K(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})).$$

Remark 2.3. In increasing generality, K-theory has been defined in the following setups:
- (i) for rings R;
- (ii) for additive categories (e.g., Proj_R);
- (iii) for (small) stable categories (e.g., $\mathrm{Mod}(R)^{\omega}$);
- (iv) for dualizable categories (e.g., $\mathrm{Mod}(R)$ or $\mathrm{Mod}(R, I)$).

2.2. **Verdier duality and 6 functors.** Let $f\colon Y \to X$ be a continuous map. Then we have an adjunction
$$f^*\colon \mathrm{Shv}(X) \rightleftarrows \mathrm{Shv}(Y) : f_*.$$
If f is locally separated and locally proper (due to Schnürer and Soergel), we have another adjunction
$$f_!\colon \mathrm{Shv}(Y) \rightleftarrows \mathrm{Shv}(X) : f^!.$$
Moreover, on $\mathrm{Shv}(X)$ we have a symmetric monoidal structure $\otimes$ and an internal hom $\underline{\mathrm{Hom}}$.

Definition 2.4. A commutative algebra $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ is called *locally rigid* if:
- (i) The functor $\otimes\colon \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ admits a cocontinuous right adjoint $\Delta\colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ which is a $\mathcal{C}$-$\mathcal{C}$-bimodule map.[2]
- (ii) $\mathcal{C}$ is dualizable; equivalently there exists a counit $\mathcal{C} \to \mathrm{Sp}$ for the comultiplication Δ.

[2] Note that this is just a *condition* and not additional structure, since Δ is automatically a lax bimodule map.

We call $\mathcal{C}$ *rigid* if in addition $\mathbf{1} \in \mathcal{C}^{\omega}$.

Example 2.5. (a) Let R be a commutative ring. Then $\mathrm{Mod}(R)$ is rigid, because we have

$$\begin{array}{ccc} \mathrm{Mod}(R) \otimes \mathrm{Mod}(R) & \xrightarrow{\otimes_R} & \mathrm{Mod}(R) \\ \simeq \downarrow & \nearrow \swarrow_{\mathrm{res}} & \\ \mathrm{Mod}(R \otimes_{\mathbb{S}} R), & & \end{array}$$

where the upper diagonal map is given by base-change along $m\colon R \otimes_{\mathbb{S}} R \to R$.

(b) For a homological epi $R \to R/I$ of commutative rings, the category $\mathrm{Mod}(R, I)$ is locally rigid, and it is rigid if and only if I is compact as an R-module.

(c) A small stable category $\mathcal{C}$ is rigid if and only if $\mathrm{Ind}(\mathcal{C})$ is rigid.

(d) Let $X \in \mathrm{LCHaus}$. Then $\mathrm{Shv}(X)$ is locally rigid; if moreover X is compact, then $\mathrm{Shv}(X)$ is rigid. To see this, note that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Shv}(X) \otimes \mathrm{Shv}(X) & \overset{\otimes}{\searrow} & \\ \simeq \downarrow & & \\ \mathrm{Shv}(X \times X) & \underset{\Delta_*}{\overset{\Delta^*}{\rightleftarrows}} & \mathrm{Shv}(X) \end{array}$$

and note that $\Delta_* = \Delta_!$, so that Δ_* has a right adjoint. The Frobenius identity (that is, the fact that Δ_* is a bimodule map) follows from the projection formula.

The counit is given by $\Gamma_c = p_!\colon \mathrm{Shv}(X) \to \mathrm{Sp}$, where $p\colon X \to \mathrm{pt}$ is the tautological map.

(e) The category $D(\mathbb{Z})_p^\wedge$ is compactly generated and locally rigid. But it is not rigid, because the unit $\mathbb{Z}$ is not compact.

Proposition 2.6. *(1) If $\mathcal{C}$ is locally rigid, then*

$$\mathrm{Sp} \xrightarrow{\mathrm{unit}} \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C}$$

exhibits $\mathcal{C}$ as a Frobenius algebra (i.e., the composition is the coevaluation for a self-duality on $\mathcal{C}$). In particular, $\mathcal{C} \simeq \mathcal{C}^\vee$.

(2) The counit $\mathcal{C} \to \mathrm{Sp}$ (which is dual to the unit $\mathrm{Sp} \to \mathcal{C}$) is equivalent to

$$\begin{aligned} \mathcal{C} &\to \mathrm{Sp}, \\ X &\mapsto \mathrm{map}^{\mathrm{ca}}(\mathbf{1}, -) \simeq \Gamma_c(-). \end{aligned}$$

The self-duality is exhibited by the equivalence

$$\begin{aligned} \mathcal{C} &\xrightarrow{\sim} \mathrm{Fun}^L(\mathcal{C}, \mathrm{Sp}), \\ X &\mapsto \mathrm{map}^{\mathrm{ca}}(\mathbf{1}, X \otimes -), \\ (F \otimes \mathrm{id})(\Delta(\mathbf{1})) &\hookleftarrow F. \end{aligned}$$

Example 2.7. We have

$$\mathrm{Shv}(X) \simeq \mathrm{Shv}(X)^\vee \simeq \mathrm{coShv}(X),$$

which is known as Verdier duality. The evaluation map for this duality is given by

$$\begin{aligned} \mathrm{Shv}(X) \otimes \mathrm{Shv}(X) &\to \mathrm{Sp}, \\ (\mathcal{F}, \mathcal{G}) &\mapsto \Gamma_c(\mathcal{F} \otimes \mathcal{G}). \end{aligned}$$

For a map $f\colon Y \to X$ in LCHaus, the functor $f_!$ is dual to f^*.

Proposition 2.8. *Let $\mathcal{C}$ be locally rigid and $\mathcal{M}$ a $\mathcal{C}$-module (in $\mathrm{Pr}^L_{\mathrm{st}}$). Then $\mathcal{M}$ is dualizable relative to $\mathcal{C}$ (i.e., dualizable in $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^L_{\mathrm{st}})$) if and only if $\mathcal{M}$ is dualizable in $\mathrm{Pr}^L_{\mathrm{st}}$.*

Example 2.9. A $\mathbb{Z}$-linear stable category is dualizable over $\mathbb{Z}$ if and only if it is dualizable over $\mathbb{S}$.

Definition 2.10. A morphism $f\colon x \to y$ in a closed symmetric monoidal category $\mathcal{C}$ is called *trace class* if it lifts as follows:

$$\begin{array}{ccc} & & \underline{\mathrm{Hom}}(x, \mathbf{1}) \otimes y \\ & \overset{\exists}{\nearrow} & \downarrow \\ \mathbf{1} & \xrightarrow[f]{} & \underline{\mathrm{Hom}}(x, y). \end{array}$$

Theorem 2.11 (Clausen, Ramzi, Scholze)**.** *Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$, such that the underlying category is dualizable, then:*

(a) $\mathcal{C}$ is locally rigid if and only if

$$\{\textit{compact morphisms}\} \subseteq \{\textit{trace class morphisms}\}.$$

(b) The unit $\mathbf{1} \in \mathcal{C}$ is compact if and only if

$$\{\textit{trace class morphisms}\} \subseteq \{\textit{compact morphisms}\}.$$

(c) $\mathcal{C}$ is rigid if and only if the classes of compact morphisms and of trace class morphisms agree.

Example 2.12. In order to see that $\mathrm{Shv}(X)$ is locally rigid, it thus suffices to see that the maps $\Sigma^\infty_+ \underline{U} \to \Sigma^\infty_+ \underline{V}$ are trace class for all $U \Subset V$.

Definition 2.13. Let $\mathcal{A} \to \mathcal{B}$ be a map in $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$. Then $\mathcal{B}$ is called *locally rigid over $\mathcal{A}$* if:

(i) The multiplication $\mathcal{B} \otimes_{\mathcal{A}} B \to \mathcal{B}$ has an $\mathcal{A}$-linear and cocontinuous right adjoint $\Delta\colon \mathcal{B} \to \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$ which is a $\mathcal{B}$-$\mathcal{B}$-bimodule map.

(ii) $\mathcal{B}$ is dualizable relative to $\mathcal{A}$ (i.e., $\mathcal{B}$ is dualizable in $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L_{\mathrm{st}})$); equivalently, there exists a counit $\mathcal{B} \to \mathcal{A}$ for the comultiplication Δ.

Example 2.14. Let $f\colon Y \to X$ be a locally proper and separated map of topological spaces. Then $\mathrm{Shv}(Y)$ is locally rigid over $\mathrm{Shv}(X)$.

Proposition 2.15. *Let $\mathcal{A} \to \mathcal{B}$ be locally rigid.*

(a) A $\mathcal{B}$-module $\mathcal{M}$ is dualizable over $\mathcal{B}$ if and only if $\mathcal{M}$ is dualizable over $\mathcal{A}$.

(b) Given an algebra map $\mathcal{B} \to \mathcal{C}$, then $\mathcal{C}$ is locally rigid relative to $\mathcal{B}$ if and only if $\mathcal{C}$ is locally rigid relative to $\mathcal{A}$.

Theorem 2.16. *The category $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})^{\mathrm{op}}$ carries a 6-functor formalism in which the exceptional maps are locally rigid maps.*

3. Talk 3

Definition 3.1. A *simple anima* is a compact anima Z together with a lift

$$\begin{array}{ccccc} \chi^{\mathrm{loc}} \in A(\mathrm{pt}) \otimes Z & \xrightarrow{\text{Assembly}} & A(Z) & \ni & \chi = [\mathbb{S}] \\ & & \downarrow \simeq & & \\ & & K((\mathrm{Sp}^Z)^\omega) & = & K(\mathbb{S}[\Omega Z]) \end{array}$$

The ∞-groupoid $\mathrm{Ani}^{\mathrm{simple}}$ is defined as the pullback

$$\begin{array}{ccc} \mathrm{Ani}^{\mathrm{simple}} & \longrightarrow & \mathrm{Ani}_{*/} \\ \downarrow & \lrcorner & \downarrow \\ (\mathrm{Ani}^{\omega})^{\simeq} & \longrightarrow & \mathrm{Ani}, \end{array}$$

where the bottom map is given by mapping a compact anima χ to the anima of lifts χ^{loc} of χ.

Theorem 3.2 (Wall '65)**.** *A compact anima is a finite anima if and only if it refines to a simple anima.*

Theorem 3.3 (Whitehead '50)**.** *A homotopy equivalence between finite CW complexes is simple (i.e., homotopic to a composition of elementary expansion and collapse maps) if and only if it refines to a map in* $\mathrm{Ani}^{\mathrm{simple}}$.

Theorem 3.4 (Hatcher, Waldhausen, Waldhausen–Jahren–Rognes)**.** *The ∞-groupoid* $\mathrm{Ani}^{\mathrm{simple}}$ *is equivalent to Hatcher's classifying space of simple homotopy types, i.e., the geometric realization*

$$|\{\mathrm{Polyhedra}, \text{simple maps}\}| \simeq |\{\mathrm{sSet}^{\mathrm{fin}}_{\mathrm{nd}}, \text{simple maps}\}|.$$

Theorem 3.5.
1. *West '77: Every compact manifold (AMR) has the homotopy type of a finite CW complex.*
2. *Chapman '65: Every homeomorphism between finite CW complexes is a simple homotopy equivalence.*

We construct a functor

$$\left\{\begin{matrix}\text{nice compact topological}\\ \text{spaces with homeomorphisms}\end{matrix}\right\} \to \mathrm{Ani}^{\mathrm{simple}}.$$

Recall: Let X be a locally compact Hausdorff space. We have seen that $\mathrm{Shv}(X;\mathrm{Sp})$ is dualizable.

Definition 3.6. Let $\mathcal{C}$ be dualizable. We define

$$\widehat{\mathrm{coShv}}(X;\mathcal{C}) := \underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Shv}(X);\mathcal{C}) \subseteq \mathrm{Ind}(\mathrm{coShv}(X;\mathcal{C})).$$

It is covariantly functorial in proper maps $f\colon X \to Y$ induced by $f^*\colon \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$.

Moreover, we define

$$\widehat{\mathrm{coShv}}_{\mathrm{cs}}(X;\mathcal{C}) = \operatorname*{colim}_{\substack{K\subseteq X\\ \text{compact}}} \widehat{\mathrm{coShv}}(K;\mathcal{C}),$$

which is functorial in all maps.

Remark 3.7. We have

$$\big(\widehat{\mathrm{coShv}}(X;\mathcal{C})\big)^{\omega} = \mathrm{Fun}^{sL}\big(\mathrm{Sp}, \widehat{\mathrm{coShv}}(X;\mathcal{C})\big) = \mathrm{Fun}^{sL}(\mathrm{Shv}(X);\mathcal{C}) \subseteq \mathrm{coShv}(X;\mathcal{C}),$$

which is a full subcategory on all cosheaves $\mathcal{F}$ such that $\mathcal{F}(U) \to \mathcal{F}(V)$ is compact for $U \Subset V$.

Proposition 3.8. *Assume that the topos* $\mathrm{Shv}(X;\mathrm{Ani})$ *is of locally constant shape (e.g., if X is hypercomplete and sublocally contractible, or is ANR). Equivalently, the functor $p^*\colon \mathrm{Ani} \to$* $\mathrm{Shv}(X, \mathrm{Ani})$ *has a left adjoint $p_\natural$ (in addition to its obvious right adjoint p_*).*

Then $\mathrm{Shv}(X;\mathrm{Sp})$ *is proper. If X is countable at ∞, then it is ω_1-compact.*

Proof. We need to show that the evaluation

$$\mathrm{Shv}(X;\mathrm{Sp})\otimes\mathrm{Shv}(X;\mathrm{Sp})\xrightarrow{\otimes=\Delta^*}\mathrm{Shv}(X;\mathrm{Sp})\xrightarrow{p_!}\mathrm{Sp}$$

is strongly left adjoint. Since Δ^* is strongly left adjoint, we have to show that $p_!$ is strongly left adjoint. As $p_!$ is dual to p^*, this is the case if and only if p^* admits a left adjoint. □

Corollary 3.9. *Under these assumptions we have*

$$\begin{aligned} K^{\mathrm{cont}}(\widehat{\mathrm{coShv}}(X;\mathcal{C})) &= KK^{\mathrm{cont}}(\mathrm{Shv}(X);\mathcal{C}) \overset{def}{=} \mathrm{map}_{\mathrm{Mot}}(\mathcal{U}\,\mathrm{Shv}(X),\mathcal{U}\mathcal{C}) \\ &\cong \mathrm{H}_{\mathrm{lf}}(X;K^{\mathrm{cont}}(\mathcal{C})) = p_*p^!K^{\mathrm{cont}}(\mathcal{C}) \\ &\cong \Pi_\infty X\otimes K^{\mathrm{cont}}(\mathcal{C}) \qquad \textit{(if } X \textit{ is compact)} \end{aligned}$$

where $\mathcal{U}$ is the universal localizing invariant and $\Pi_\infty X$ denotes the shape of X.

Proposition 3.10. *Let $X\in\mathrm{LCHaus}$ be σ-compact and of stably locally constant shape. There is a canonical compact object*

$$\chi^{\mathrm{loc}}\in\widehat{\mathrm{coShv}}(X;\mathrm{Sp})$$

given as $p_\natural = p_!(-\otimes\omega_X)\colon \mathrm{Shv}(X;\mathrm{Sp})\to\mathrm{Sp}$. As a cosheaf it is given by $U\mapsto\Sigma^\infty_+\Pi_\infty U$.

Corollary 3.11. *If X is compact, then $\chi^{\mathrm{loc}}\in\Pi_\infty X\otimes K^{\mathrm{cont}}\,\mathrm{Sp}$.*

Question 3.12. Is there a "nice" description of $K_0^{\mathrm{cont}}(\mathcal{C})$?

Theorem 3.13 (Bartels–N.)**.** *There is a strongly left adjoint functor*

$$A\colon \widehat{\mathrm{coShv}}_{\mathrm{cs}}(X;\mathcal{C})\to\mathcal{C}^{\Pi_\infty X}=\mathrm{Loc}(X;\mathcal{C})$$

with the following properties:

(i) *It induces the assembly map on K-theory.*
(ii) *It takes χ^{loc} to $\chi=\mathbb{S}$ if X is compact and $\mathcal{C}=\mathrm{Sp}$.*

Proof/Construction. We prove (i) under the assumption that X is compact. Then

$$\begin{aligned} \widehat{\mathrm{coShv}}(X;\mathcal{C}) = \underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Shv}(X);\mathcal{C}) &\xrightarrow{\otimes\,\mathrm{Sp}^{\Pi_\infty X}} \underline{\mathrm{Hom}}\big(\mathrm{Shv}(X)\otimes\mathrm{Sp}^{\Pi_\infty X},\mathcal{C}^{\Pi_\infty X}\big) \\ &\xrightarrow{D^*} \underline{\mathrm{Hom}}^{\mathrm{dual}}(\mathrm{Sp},\mathcal{C}^{\Pi_\infty X}) = \mathcal{C}^{\Pi_\infty X}, \end{aligned}$$

where $D\colon \mathrm{Sp}\to\mathrm{Shv}(X)\otimes\mathrm{Sp}^{\Pi_\infty X}$ is left adjoint to

$$\mathrm{Shv}(X)\otimes\mathrm{Sp}^{\Pi_\infty X}\xrightarrow{\psi^*}\mathrm{Shv}(X)\otimes\mathrm{Shv}(X)\xrightarrow{\Delta^*}\mathrm{Shv}(X)\to\mathrm{Sp}\,.$$

□