EXTENDED NOTES TO ACCOMPANY THE LECTURES ON "TACKLING RANK CONJECTURES USING ADAMS OPERATIONS"

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Contents

1. Rank conjectures in algebra (and topology)

1.1. **Notation and conventions.** Usually, we will be working over a local ring $(R, \mathfrak{m}, k)$. The notation means R is a commutative, Noetherian ring with a unique maximal ideal $\mathfrak{m}$, and we set $k := R/\mathfrak{m}$. Later, in a few spots, in order to relate the ideas presented here to topology, we will talk about modules (in fact, dg modules) over graded polynomial rings.

Example 1.1. $R = (k[x_1, \dots, x_n]/I)_\mathfrak{m}$ for some field k and ideal $I = (f_1, \dots, f_c)$ generated by polynomials $f_i \in \mathfrak{m}$ where $\mathfrak{m} = (x_1, \dots, x_n)$, or the completion of such: $k[[x_1, \dots, x_n]]/(f_1, \dots, f_c)$. We identify k with $R/\mathfrak{m}$.

A chain complex of R-modules will almost always be indexed by subscripts, with its differential, usually written as ∂, lowering the degree by one:

$$\mathbb{M} = (\cdots \xrightarrow{\partial} M_n \xrightarrow{\partial} M_{n-1} \xrightarrow{\partial} \cdots)$$

On rare occasions we use superscripts — in this case, M^n should be interpreted as M_{-n} and the differential will increase the superscript by one. $H_i(\mathbb{M})$ denotes the i-th homology module of a complex, $H_i(\mathbb{M}) = \frac{\ker(M_i \to M_{i-1})}{\operatorname{im}(M_{i+1} \to M_i)}$, and $H(\mathbb{M})$ denotes the graded R-module $H(\mathbb{M}) := \bigoplus_i H_i(M)$.

Given two complexes $\mathbb{M}$ and $\mathbb{N}$ of R-modules, we define their *tensor product* $\mathbb{M} \otimes_R \mathbb{N}$ to be the complex obtained by totalization of the evident bicomplex; in more detail

$$(\mathbb{M} \otimes_R \mathbb{N})_l = \bigoplus_{i+j=l} M_i \otimes_R N_j$$

equipped with the differential given by

$$\partial(m \otimes n) = \partial(m) \otimes n + (-1)^i m \otimes \partial(n) \text{ for } m \in M_i \text{ and } n \in N_j. \tag{1.2}$$

For elements $f_1, \dots, f_c \in R$, the *Koszul complex* $\mathrm{Kos}_R(f_1, \dots, f_c)$ is defined as follows: For $c = 1$,

$$\mathrm{Kos}_R(f_1) = (0 \to R \xrightarrow{f_1} R \to 0) \text{ with } R \text{ in degrees 0 and 1,}$$

and in general

$$\mathrm{Kos}_R(f_1, \dots, f_c) = \mathrm{Kos}_R(f_1) \otimes_R \cdots \otimes_R \mathrm{Kos}_R(f_c).$$

A better way of describing $\mathrm{Kos}_R(f_1, \dots, f_c)$ is as the exterior algebra $\Lambda_R(e_1, \dots, e_c)$ over R on degree one generators $e_1, \dots, e_c$ equipped with the unique R-linear differential such that $\partial(e_i) = f_i$ and ∂ obeys the graded Leibniz rule: $\partial(\alpha \cdot \beta) = \partial(\alpha) \cdot \beta + (-1)^{|\alpha|} \alpha \cdot \partial(\beta)$ for all $\alpha, \beta \in \Lambda_R(e_1, \dots, e_c)$.

For instance,

$$\mathrm{Kos}_R(x, y, z) = \left(0 \to R \xrightarrow{C} R^3 \xrightarrow{B} R^3 \xrightarrow{A} R \to 0\right)$$

where

$$C = \begin{bmatrix} z \\ -y \\ x \end{bmatrix}, \; B = \begin{bmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{bmatrix}, \; A = \begin{bmatrix} x & y & z \end{bmatrix}.$$

Here, the matrices are relative to the following bases: $e_1 \wedge e_2 \wedge e_3$ for F_3, $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ for F_2, $\{e_1, e_2, e_3\}$ for F_1, and 1 for F_0.

1.2. **Some commutative, and homological, algebra over local rings.** Fix a local ring $(R, \mathfrak{m}, k)$.

A sequence of elements $x_1, \dots, x_c \in \mathfrak{m} \subseteq R$ contained in the maximal ideal of R is a *regular sequence* if x_{i+1} is a non-zero divisor in the quotient ring $R/(x_1, \dots, x_i)$ for all $0 \le i \le c-1$. This is equivalent to the condition that the Koszul complex $\mathrm{Kos}_R(x_1, \dots, x_c)$ is exact everywhere except in degree 0, and hence is a bounded free resolution of $M := R/(x_1, \dots, x_c)$.

Exercise 1.3. Prove the equivalence asserted above. *Tip*: Use that $\mathrm{Kos}_R(x_1, \dots, x_{i+1})$ is the mapping cone of $\mathrm{Kos}_R(x_1, \dots, x_i) \xrightarrow{x_{i+1}} \mathrm{Kos}_R(x_1, \dots, x_i)$.

The *depth* of R, written $\operatorname{depth}(R)$, is the largest value of c such that a regular sequence $x_1, \ldots, x_c \in \mathfrak{m}$ exists. Since $\dim(R/x) = \dim(R) - 1$ for a non-zero-divisor $x \in \mathfrak{m}$, it follows that

$$\operatorname{depth}(R) \leq \dim(R);$$

when equality holds, R is called a *Cohen-Macaulay (CM)* ring.

Recall that a local ring is regular if $\mathfrak{m}$ can be generated by $d = \dim(R)$ elements, say $x_1, \ldots, x_d$. In this case, $x_1, \ldots, x_d$ is necessarily a regular sequence. In particular, regular rings are CM. Note that in this case $\operatorname{Kos}_R(x_1, \ldots, x_d)$ is the minimal resolution of k.

Recall that an R-module M has *finite length* if it admits a filtration

$$0 = M_0 \subseteq \cdots \subseteq M_l = M$$

with M_i/M_{i-1} simple (in the local case, this means isomorphic to $k = R/\mathfrak{m}$) for all i; the value of l, which is an invariant, is the *length* of M, written as $\operatorname{length}_R(M)$. When R "contains its residue field" (i.e. R is a k-algebra such that the composition of the canonical maps $k \to R \twoheadrightarrow R/\mathfrak{m} = k$ is an isomorphism — for example $R = k[[x_1, \ldots, x_n]]/I$), we have $\operatorname{length}_R(M) = \dim_k(M)$.

For a local ring R, given a complex of R-modules

$$\mathbb{F} = (\cdots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots)$$

we say $\mathbb{F}$

(1) is *minimal* if each F_i is free of finite rank, and $\partial_i(F_i) \subseteq \mathfrak{m} \cdot F_{i-1}$ for all i,
(2) is *finite free* if it is bounded ($F_m = 0$ for $m \ll 0$ and $m \gg 0$) and each F_i is free of finite rank,
(3) has *finite length homology* if $\operatorname{length} H_i(\mathbb{F}) < \infty$ for all i,
(4) is *tiny* (this is non-standard notation) if $\mathbb{F}$ is finite free with finite length homology and it is of the form $\mathbb{F} = (0 \to F_d \to \cdots F_0 \to 0)$ where d is the Krull dimension of R.

Example 1.4. Let $x_1, \ldots, x_d \in \mathfrak{m} \subseteq R$ be a system of parameters — recall that this means $d = \dim(R)$ and $R/(x_1, \ldots, x_d)$ has finite length. Then $\mathbb{F} = \operatorname{Kos}_R(x_1, \ldots, x_d)$ is a tiny complex. This holds since the homology $\mathbb{F}$ is a finitely generated module over $H_0(\mathbb{F}) = R/(x_1, \cdots, x_d)$, which has finite length.

Theorem 1.5. *Let $(R, \mathfrak{m}, k)$ be local of Krull dimension d and assume $\mathbb{F}$ is a finite free complex.*

(1) *If $\mathbb{F}$ has finite length homology and is of the form $\mathbb{F} = (0 \to F_s \to \cdots \to F_0 \to 0)$ and is non-trivial (i.e., $H(F) \neq 0$), then $s \geq d$. This is the New Intersection Theorem* [Rob89].
(2) *If $\mathbb{F}$ is tiny and non-trivial, then the non-zero homology of $\mathbb{F}$ is precisely in the range $[0, \dim(R) - \operatorname{depth}(R)]$ — i.e., $H_i(\mathbb{F}) = 0$ for all $i > \dim(R) - \operatorname{depth}(R)$ and $H_i(\mathbb{F}) \neq 0$ for all $0 \leq i \leq \dim(R) - \operatorname{depth}(R)$. In particular, if R is CM then $H_i(\mathbb{F}) = 0$ for all $i \neq 0$ and hence $\mathbb{F}$ is a resolution of the module $H_0(\mathbb{F})$.*
(3) *Conversely, if M is non-zero R-module having finite length and finite projective dimension, then R must be CM and the minimal resolution of M is tiny. This is a consequence of the Auslander-Buchsbaum Formula ($\operatorname{pd}_R(M) = \operatorname{depth}(R) - \operatorname{depth}(M)$) and the New Intersection Theorem.*

So, there is a dichotomy involving tiny complexes: When R is CM, every tiny complex is the resolution of a module (necessarily of finite length and finite projective dimension). When R is not CM, there are no such modules, and every non-trivial tiny complex has homology in strictly positive degrees.

Example 1.6. Let $x_1, \dots, x_d \in \mathfrak{m}$ be a system of parameters for R. R is CM if and only if $x_1, \dots, x_d$ is a regular sequence if and only if the tiny complex $\mathrm{Kos}_R(x_1, \dots, x_d)$ is the resolution of a module (namely, $R/(x_1, \dots, x_d)$).

1.3. **Algebraic Rank Conjectures.**

Conjecture 1.7 (Buchsbaum-Eisenbud-Horrocks (BEH) Conjecture)**.** *Assume* $(R, \mathfrak{m}, k)$ *is a local ring of dimension d and M is a non-zero module of finite length and finite projective dimension, and let $\mathbb{F}$ be a finite free resolution of M. Then* $\mathrm{rank}_i(F_i) \geq \binom{d}{i}$.

Status of this conjecture: More or less wide open.

Remark 1.8. This is sometimes phrased in terms of Betti numbers: The *i-th Betti number* of M is $\mathrm{rank}(F_i)$ where $\mathbb{F}$ is the minimal free resolution of M. Equivalently, $\beta_i(M) = \dim_k \mathrm{Tor}_i^R(M, k) = \dim_k \mathrm{Ext}_R^i(M, k)$. With this notation, the BEH conjecture is $\beta_i(M) \geq \binom{d}{i}$ whenever M has finite length and finite projective dimension. As Craig Huneke has pointed out to me, there does not seem to be a compelling reason to include "finite projective dimension" as an assumption in this conjecture.

As partial motivation:

Example 1.9. Suppose R is Cohen-Macaulay and $x_1, \dots, x_d \in \mathfrak{m}$ is a regular sequence, where $d = \dim(R)$. Then $R/(x_1, \dots, x_d)$ has finite length and finite projective dimension, and its minimal free resolution is given by $\mathbb{F} = \mathrm{Kos}_R(f_1, \dots, f_d)$, which satisfies $\mathrm{rank}_R F_i = \binom{d}{i}$. Thus, the lower bound in BEH, if correct, is sharp.

More generally we have:

Proposition 1.10 (Buchsbaum-Eisenbud)**.** [BE77] *If $M = R/I$ has finite length and finite projective dimension and its minimal free resolution $\mathbb{F}$ admits the structure of commutative dga, then there exists an injection $\mathbb{K} \hookrightarrow \mathbb{F}$ for some Koszul complex $\mathbb{K}$ on a system of parameters, and hence* $\mathrm{rank}\, F_i \geq \mathrm{rank}\, K_i = \binom{d}{i}$.

Remark 1.11. The BEH Conjecture appears in [BE77] and [Har79]. For the latter, it is actually phrased as a question. For the former, the original conjecture was that the hypothesis of Proposition 1.10 holds for any (cyclic) module of finite length and finite projective dimension. Counter-examples to this stronger conjecture were already known by Avramov (see [Avr81]), but the conjecture now known as BEH has survived.

I will say nothing more about the BEH Conjecture. Instead, the focus of these talks will be on the following conjecture. Some refer to it the "Weak BEH" conjecture, but I prefer calling it the Total Rank Conjecture.

Conjecture 1.12 (Total Rank Conjecture)**.** *Assume* $(R, \mathfrak{m}, k)$ *is a local ring of dimension d.*

- *(Avramov) If M is a non-zero module of finite length and finite projective dimension (and hence R is CM), and $\mathbb{F}$ is a finite free resolution of M, then* $\operatorname{rank}(\mathbb{F}) = \sum_i \mathit{rank}(F_i) \geq 2^d$.
- *More generally, if $\mathbb{F}$ is a non-trivial tiny complex, then* $\operatorname{rank}(\mathbb{F}) \geq 2^d$. *(This was not conjectured by Avramov, but should have been.)*

Status of this conjecture: It is know when R is equi-characteristic (i.e., contains a field as a subring) and for certain rings of mixed characteristic so long as $\operatorname{char}(k) \neq 2$.

Example 1.13. If R is any local ring (not necessarily CM) and $x_1, \dots, x_d \in \mathfrak{m}$, where $d = \dim(R)$, is any system of parameters, then $\mathbb{F} := \operatorname{Kos}_R(f_1, \dots, f_d)$ is a tiny complex meeting the predicted lower bound.

Conjecture 1.14 (Total Rank Conjecture for Finite Free Complexes (Folklore))**.** *If R is a local ring of dimension d and $\mathbb{F}$ is any non-trivial finite free complex with finite length homology, then* $\operatorname{rank}(\mathbb{F}) \geq 2^d$.

Status of this conjecture: Known to be false when $\operatorname{char}(k) \neq 2$. The case $\operatorname{char}(k) = 2$ remains open and is intriguing.

Remark 1.15. Every finite free complex $\mathbb{F}$ decomposes as $\mathbb{F} = \overline{\mathbb{F}} \oplus \mathbb{F}_{\text{trivial}}$ with $\overline{\mathbb{F}}$ minimal and $\mathbb{F}_{\text{trivial}}$ exact (and hence contractible). In particular, $\beta_i(\mathbb{F}) = \operatorname{rank}(\overline{F}_i)$, and the assertion that $\operatorname{rank}(\mathbb{F}) \geq 2^d$ for all suitable $\mathbb{F}$ is equivalent to the assertion that $\beta(\mathbb{F}) \geq 2^d$ for all suitable $\mathbb{F}$.

1.4. **Generalization to dg modules, a brief aside on the toral rank conjectures.** Let S be the graded ring $S = k[x_1, \dots, x_d]$ with each x_i declared to be of homological degree -2. Set $\mathfrak{m} = (x_1, \dots, x_d)$ and $k = S/\mathfrak{m}$. A *dg S-module* $\mathbb{M}$ is a graded S-module equipped with an S-linear endomorphism ∂ of degree -1 such that $\partial^2 = 0$. For instance k is a dg-S-module with trivial differenital. A dg-module $\mathbb{F}$ is *semi-free* if it admits a filtration

$$0 = \mathbb{F}^{(-1)} \subseteq \mathbb{F}^{(0)} \subseteq \mathbb{F}^{(1)} \subseteq \cdots \mathbb{F}$$

such that $\mathbb{F} = \bigcup_i \mathbb{F}^{(i)}$ and $\mathbb{F}^{(i)}/\mathbb{F}^{(i-1)}$ is isomorphic to a direct sum of shifts of S with trivial differential.

Conjecture 1.16 (Total Rank Conjecture for dg Modules)**.** *If $\mathbb{F}$ is a semi-free dg S-module with non-zero, finite length homology then* $\operatorname{rank}(\mathbb{F}) \geq 2^d$.

Status of this conjecture: Known to be false when $\operatorname{char}(k) \neq 2$; open when $\operatorname{char}(k) = 2$.

Conjecture 1.17 (Total Rank Conjecture for Formal dg Modules)**.** *If $\mathbb{F}$ is a semi-free dg S-module with non-zero, finite length homology that is quasi-isomorphic, as a dg-S-module, to its homology, then* $\operatorname{rank}(\mathbb{F}) \geq 2^d$.

Status of this conjecture: Known to hold for any field k.

Let me indicate briefly, and rather heuristically, how this algebraic conjecture is related to topology — Leo will go into more details. First, I recall:

Conjecture 1.18 (Toral Rank Conjecture)**.** *Suppose the d-dimensional torus $T = T_d = (S^1)^{\times d}$ acts freely (and "reasonably") on a compact CW complex X. Then* $\operatorname{rank}_{\mathbb{Q}} H^*_{\text{sing}}(X, \mathbb{Q}) \geq 2^d$, *where H^*_{sing} denotes the singular cohomology of a topological space.*

Status of this conjecture: Open

Under the assumption of the Toral Rank Conjecture, set $Y = X/T$, the quotient space for the action. Then there is an induced map $p : Y \to BT$ where BT is the classifying space of the torus. We have $BT = (\mathbb{CP}^\infty)^{\times d}$ and the rational cohomology ring of BT may be identified with the graded ring S above, with $k = \mathbb{Q}$. This map leads to a dg-S-module (in fact, dg S-*algebra*) $\mathcal{A}(Y)$ with the following two properties:

- $H^*(\mathcal{A}(Y)) \cong H^*_{\text{sing}}(Y, \mathbb{Q})$ and
- $H^*(\mathcal{A}(Y) \otimes^{\mathbb{L}}_S k) \cong H^*(X, \mathbb{Q})$.

Example 1.19. Taking $X = T_d$ with the action given by multiplication. Then $H^i_{\text{sing}}(X, \mathbb{Q}) = \binom{d}{i}$, attaining the predicted lower bound of the Toral Rank Conjecture. In this example, we have $Y = \{pt\}$ and thus $\mathcal{A}(Y) = k$. We may resolve k by the semi-free dg-S-module $\mathrm{Kos}_S(x_1, \dots, x_d)$ and hence $\mathcal{A}(Y) \otimes^{\mathbb{L}}_S k$ is the exterior algebra over k generated by d elements of (cohomological) degree 1 with trivial differential. This confirms that $\mathcal{A}(Y) \otimes^{\mathbb{L}}_S k$ gives the cohomology of X.

So, the example of the self-action of the torus is a topological version of the example of k being a module of finite length and finite projective dimension over a regular local ring.

Proposition 1.20. *The Total Rank Conjecture for dg Modules implies the Toral Rank Conjecture.*

Proof. Choose a minimal, semi-free dg S-module $\mathbb{F}$ quasi-isomorphic to $\mathcal{A}(Y)$. Then

$$\operatorname{rank}(\mathbb{F}) = \beta(\mathbb{F}) = \dim_k H^*(\mathcal{A}(Y) \otimes^{\mathbb{L}}_S k) = \dim_k H^*_{\text{sing}}(X, \mathbb{Q}),$$

and the claim is now clear. □

As noted, the TRC for dg Modules is false in characteristic 0, but the Toral Rank Conjecture remains open. A key point is that the counter-examples of the former are *not* dgas.

1.4.1. *An aside on BGG.* Let me mention that construction $Y = X/T$ admits an algebraic manifestation. Let $E = \Lambda_{\mathbb{Q}}(e_1, \dots, e_d)$ be an exterior algebra on d generators of (homological) degree 1. Then the *Bernsetien-Gel'fand-Gel'fand (BGG) correspondence* relates dg-S-modules and dg-E-modules. Specifically, if $\mathbb{M}$ is a dg-S-module, then $\mathbb{M} \otimes_k E$ equipped with the differential $\partial_{\mathbb{M}} \otimes \mathrm{id} + \sum_i x_i \otimes e_i$ is a dg-E-module. For example, if $\mathbb{M} = k$ equipped with the trivial differential, the corresponding dg-E-module is E equipped with the trivial differential. More generally, under this correspondence, a dg-S-module $\mathbb{M}$ with finite length total homology corresponds to a perfect dg-E-module $\mathbb{P}$, such that $\sum_i \beta_i(\mathbb{M}) = \dim_k H(\mathbb{P})$. Thus:

Proposition 1.21. *The TRC for dg Modules is equivalent to the conjecture that every non-trivial perfect dg-E-module $\mathbb{P}$ satisfies* $\dim_k H(\mathbb{P}) \geq 2^d$.

Remark 1.22. Under BGG, formal dg S-modules correspond to dg-E-modules having linear differentials.

1.5. **Another family of (algebraic and topological) conjectures.** Let's start with topology this time. For a prime p define the *n-dimensional p-torus* to be the group $(C_p)^{\times n}$ where C_p is cyclic of order p. The name comes from the fact that this group forms the p-torsion of n-torus T_n. Ergun will go into much more detail on the following:

Conjecture 1.23 (Carlsson)**.** *If a space X admits a cellular and free action by a n-dimensional p-torus, then $\sum_i \dim_{\mathbb{Z}/p} H_i(X;\mathbb{Z}/p) \geq 2^n$.*

Status of this conjecture: Open

Such an action determines a finite free complex $\mathbb{F}$ over the finite dimensional $\mathbb{Z}/p$-algebra $\mathbb{Z}/p[x_1,\dots,x_n]/(x_1^p,\dots,x_n^p)$ with $H_i(\mathbb{F}) \cong H_i(X,\mathbb{Z}/p)$. This leads to:

Conjecture 1.24 (Algebraic Analogue of Carlsson's Conjecture [AB88])**.** *Let k be a field of characteristic $p > 0$ and let R be the ring $R = k[x_1,\dots,x_n]/(x_1^p,\dots,x_n^p)$ for any $n \geq 1$. If $\mathbb{F}$ is any non-trivial, finite free complex over F, then* $\dim_k H(\mathbb{F}) = \sum_i \dim_k H_i(\mathbb{F}) \geq 2^n$*.*

Status of this conjecture: Known to be false when $p \neq 2$; open and intriguing when $p = 2$.

Notice that this conjecture focuses on the *homology* of $\mathbb{F}$, not its Betti numbers. But it is nevertheless related to the rank conjectures introduced above, at least in the case $p = 2$. Let me explain:

When $p = 2$, we may identify R with the exterior algebra over k on d generators: $R \cong \Lambda_k(e_1,\dots,e_n)$ (with $e_i = x_i$). (Importantly, this is *not* a graded ring, but rather viewed as an ordinary ring, i.e., concentrated in degree 0.) Thus, a version of BGG (rediscovered by Carlsson) relates complexes of R-modules with dg-modules over the graded polynomial ring $P := k[t_1,\dots,t_n]$ with $\deg(t_i) = -1$. This equivalence sends a finite free complex $\mathbb{F}$ over R to a dg-P-module $\mathbb{G}$ with finite length homology, and moreover $\dim_k H(\mathbb{F}) = \sum_i \beta_i(\mathbb{G})$.

For instance, the complex with R in degree 0 and 0's elsewhere corresponds to a dg-P-module that is quasi-isomorphic to $k = P/\mathfrak{m}$.

Thus, Carlson's conjecture in characteristic two fits into the general framework: If the evident analogue of the Total Rank Conjecture for dg Modules holds for the graded ring P, then the Algebraic analogue of Carlsson's conjecture holds for $p = 2$.

2. Grothendieck groups and Adams operations

The main goal of these talks is to give details on the proofs of the various Total Rank Conjectures in certain cases. I will also discuss counter-examples. A central tool in these proofs is the notion of Adams operations on Grothendieck groups. In this section we develop this tool.

2.1. **Grothendieck groups.**

Definition 2.1. For a commutative noetherian ring R, let $K_0^{\mathrm{fl}}(R)$ be the abelian group generated by isomorphism classes of finite free complexes having finite length homology and subject to the two relations

- $[\mathbb{F}] = [\mathbb{F}']$ if $\mathbb{F}$ and $\mathbb{F}'$ are quasi-isomorphic (equivalently, homotopy equivalent) complexes and
- $[\mathbb{F}] = [\mathbb{F}'] + [\mathbb{F}'']$ if there is a short exact sequence $0 \to \mathbb{F}' \to \mathbb{F} \to \mathbb{F}'' \to 0$ of complexes.

Remark 2.2. For a Zariski closed subset Z of $\mathrm{Spec}(R)$, $K_0^Z(R)$ is the abelian group generated by bounded complexes of finitely generated, projective R-modules with homology supported on Z subject to the same two relations given above. ($\mathbb{F}$ is supported on Z if the localized complex $\mathbb{F}_\mathfrak{q}$ is exact for all $\mathfrak{q} \notin Z$.) For a local ring $(R,\mathfrak{m},k)$ we have $K_0^{fl}(R) = K_0^{\{\mathfrak{m}\}}(R)$.

Exercise 2.3. Prove $[\mathbb{F}]+\mathbb{F}'] = [\mathbb{F}\oplus\mathbb{F}']$ and $[\mathbb{F}] = -[\Sigma\mathbb{F}]$ where $\Sigma\mathbb{F}$ is the suspension of $\mathbb{F}$. In particular, conclude that every class in $K_0^{fl}(R)$ may be represented by a single complex concentrated in non-negative degrees.

Definition 2.4. For a finite free complex $\mathbb{F}$ with finite length homology, its *Euler characteristic* is

$$\chi(\mathbb{F}) = \sum_i (-1)^i \operatorname{length}_R H_i(\mathbb{F}) \in \mathbb{Z}.$$

Since $\chi(-)$ respects the two relations defining the Grothendieck group, it extends to a homomorphism

$$\chi : K_0^{\mathrm{fl}}(R) \to \mathbb{Z}.$$

Exercise 2.5. If R is a regular local ring, then $\chi : K_0^{\mathrm{fl}}(R) \xrightarrow{\cong} \mathbb{Z}$ is an isomorphism. Moreover, the element $1 \in \mathbb{Z}$ corresponds to $[\mathrm{Kos}_R(x_1, \dots, x_d)]$ for a minimal set of generators $x_1, \dots, x_d$ of $\mathfrak{m}$.

The topic of the algebraic Rank Conjectures is finite free complexes, and any such complex determines a class in $K_0^{\mathrm{fl}}(R)$. At first glace it appears hopeless to tackle this conjecture by considering classed in K_0^{fl}, since "too much information is lost". For instance, if R is regular, then the only information about $\mathbb{F}$ recorded by its class $[\mathbb{F}]$ is its Euler characteristic. (The Total Rank Conjecture for Modules was open even for regular rings for several decades, and so this is hardly a trivial case.) Nevertheless, with the help of certain extra structure, namely the *Adams operations*, the TRC can be proven at least in some cases using this approach.

We will also have need for the more classical G_0 group of a ring:

Definition 2.6. For a commutative Noetherian ring R, by $G_0(R)$ we mean the abelian group generated by isomorphism classes of finitely generated R-modules subject to the relation $[M] = [M'] + [M'']$ whenever there is a short exact sequence of the form $0 \to M' \to M \to M'' \to 0$.

Exercise 2.7. If R is a regular local ring, then $G_0(R) \cong \mathbb{Z}$, generated by the class of $[R]$.

For a finite free complex $\mathbb{F}$ with finite length homology and finitely generated module M, we set

$$\chi(\mathbb{F}, M) = \chi(\mathbb{F} \otimes_R M) = \sum_i \operatorname{length}_R H_i(\mathbb{F} \otimes_R M).$$

($\mathbb{F} \otimes_R M$ is a bounded complex with finite length homology, and so this formula makes sense.)

Exercise 2.8. Show $\chi(-,-)$ preserves the relations in each argument and thus extends to a bi-linear pairing

$$\chi(-,-) : K_0^{\mathrm{fl}}(R) \otimes_{\mathbb{Z}} G_0(R) \to \mathbb{Z},$$

which we also write as $\chi(-,-)$.

2.2. **Tensor, exterior, and symmetric powers of free modules.** Adams operations arise from the non-additive functors of exterior and symmetric powers of free modules. Let us review these notions:

For a finite free module F over R, its n-th tensor power is

$$T^n(F) = T^n_R(F) = \overbrace{F \otimes_R \cdots \otimes_R F}^{n \text{ times}},$$

which is a free module of rank $\operatorname{rank}(F)^n$.

There is an action of the symmetric group Σ_n on $T^n(F)$ given by permuting the tensor factors. The *n-th symmetric power* of F is

$$\mathrm{S}^n(F) := T^n(F)/\Sigma_n = T^n(F)/\langle v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \mid \sigma \in \Sigma_n \rangle$$

with a typical generator written as $v_1 \bullet \cdots \bullet v_n$. So, $v_1 \bullet \cdots \bullet v_n = v_{\sigma(1)} \bullet \cdots \bullet v_{\sigma(n)}$ for all $\sigma \in \Sigma_n$. Equivalently, regarding $T^n(F)$ as a (left) module over the group ring $R[S_n]$, we have

$$\mathrm{S}^n(F) = R_{\text{trivial}} \otimes_{R[S_n]} T^n(F)$$

where R_{trivial} is R equipped with the (right) Σ_n action $r \cdot \sigma = r$.

Remark 2.9. Upon choosing a basis $x_1, \dots, x_r$ of F, we may identity $\mathrm{S}^n(F)$ with the degree n part of the polynomial ring $R[x_1, \dots, x_r]$.

Likewise, the *n-th exterior power* of F is

$$\Lambda^n(F) := T^n(F)/\langle v_1 \otimes \cdots \otimes v_n \mid v_i = v_j, i \neq j \rangle$$

with a typical generator written as $v_1 \wedge \cdots \wedge v_n$. A standard trick shows that $v_1 \wedge \cdots \wedge v_n = \operatorname{sign}(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)}$ for any $\sigma \in S_n$. When 2 is a invertible in R, this latter collection of relations implies the former; that is, we have

$$\Lambda^n(F) = R_{\text{sign}} \otimes_{R[S_n]} T^n(F) \text{ when 2 is invertible,}$$

where R_{sign} is R equipped with the Σ_n action $r \cdot \sigma = \operatorname{sign}(\sigma) r$.

Remark 2.10. Upon choosing a basis $e_1, \dots, e_r$ of F, we may identity $\Lambda^n(F)$ with the degree n part of the exterior algebra $\Lambda_R(e_1, \dots, e_r)$ (where $\deg(e_i) = 1$).

We may also realize $\Lambda^n(F)$ as a submodule of $T^n(F)$ via the *anti-symmetrization map*

$$\Lambda^n(F) \hookrightarrow T^n(F), \text{ given by } v_1 \wedge \cdots \wedge v_n \mapsto \sum_{\sigma \in \Sigma_n} \operatorname{sign}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

Importantly for us, in the special case when $n = 2$, the sequence

$$0 \to \Lambda^2(F) \xrightarrow{v \wedge w \mapsto v \otimes w - w \otimes v} T^2(F) \xrightarrow{\text{can}} \mathrm{S}^2(F) \to 0 \tag{2.11}$$

is exact.

Exercise 2.12. Verify that (2.11) is exact.

2.2.1. *Polynomial functors.* T^n, S^n and Λ^n are examples of "polynomial functors of degree n". Let me explain what this means just in the cases $n = 1$ and $n = 2$.

Suppose T is a functor taking finite free R-modules to finite free R-modules that sends 0 to 0. We say

(1) T is *polynomial of degree* 1 if it is non-trivial and additive: i.e., the canonical split injection $T(F) \oplus T(F') \hookrightarrow T(F \oplus F')$, induced from the canonical inclusions $F \hookrightarrow F \oplus F'$ and $F' \hookrightarrow F \oplus F'$, is actually an isomorphism, for all finite free modules F and F'.

(2) T is *polynomial of degree* 2 if the "first cross effects" bi-functor $T'(F_1, F_2) := \mathrm{coker}(T(F_1) \oplus T(F_2) \xhookrightarrow{\text{can}} T(F_1 \oplus F_2))$ is polynomial of degree 1 (i.e., non-trivial and additive) in each argument.

E.g.,

- We have
$$T^2(F_1 \oplus F_2) \cong T^2(F_1) \oplus T^2(F_2) \oplus F_1 \otimes_R F_2 \oplus F_2 \otimes_R F_1$$
and thus
$$(T^2)'(F_1, F_2) = F_1 \otimes_R F_2 \oplus F_2 \otimes_R F_1.$$
The latter is additive in each argument, and thus T^2 is polynomial of degree 2.
- We have
$$\Lambda^2(F_1 \oplus F_2) \cong \Lambda^2(F_1) \oplus \Lambda^2(F_2) \oplus F_1 \otimes_R F_2$$
and so $\left(\Lambda^2\right)'(F_1 \oplus F_2) \cong F_1 \otimes_R F_2$ and hence Λ^2 is polynomial of degree 2.
- Similarly $\left(\mathrm{S}^2\right)'(F_1 \oplus F_2) \cong F_1 \otimes_R F_2$ and S^2 is polynomial of degree 2.

Note that the first cross-effects functors for Λ^2 and S^2 coincide, each given by $(F_1, F_2) \mapsto F_1 \otimes_R F_2$. This will be an important point in defining the (second) Adams operations.

2.2.2. *The significance of* 2. A important fact is that (2.11) is naturally split *provided* 2 *is invertible in* R, by the map $\mathrm{S}^2(F) \to T^2(F)$ sending $v \bullet w$ to $\frac{1}{2} v \otimes w + w \otimes v$ or, equivalently, by the map $T^2(V) \to \Lambda^2(V)$ sending $v \otimes w$ to $\frac{1}{2} v \wedge w$. In other words, we have a natural isomorphism

$$T^2(F) \cong \Lambda^2(F) \oplus \mathrm{S}^2(F) \text{ when 2 is invertible.} \tag{2.13}$$

No such (natural) splitting exists when $\mathrm{char}(R) = 2$. This distinction turns out to be the fundamental reason the status of the various Rank Conjectures is so different depending on whether $\mathrm{char}(k) = 2$.

Moreover, when $\mathrm{char}(R) = 2$ (i.e., $1 + 1 = 0$ in R), we loose (2.13) but gain an interesting short exact sequence: note that $v_1 \wedge \cdots \wedge v_n = \mathrm{sign}(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)}$ becomes $v_1 \wedge \cdots \wedge v_n = v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)}$ since $2 = 0$ implies $1 = -1$. It follows that there is a canonical surjection

$$\mathrm{S}^n(F) \twoheadrightarrow \Lambda^n(F), \text{ given by } v_1 \bullet \cdots \bullet v_n \mapsto v_1 \wedge \cdots \wedge v_n.$$

This map is not an isomorphism since products with repeated factors are trivial in Λ^n but not typically in S^n. When $n = 2$ we can identify the kernel. Let $\mathrm{Frob}(F)$ denote applying extension of scalars along the Frobenius ring map $R \xrightarrow{r \mapsto r^2} R$; i.e., $\mathrm{Frob}(F) = R' \otimes_R F$ where $R' = R$ but viewed as an R-algebra via the Frobenius map. So, $\mathrm{Frob}(F)$ is also free, of the same rank as F, but a R-map from $\mathrm{Frob}(F)$ to an arbitrary R-module M is given by an additive map $g : F \to M$ such that $g(rv) = r^p g(v)$ for all $r \in R, v \in F$. (Intuitively, you may think of $\mathrm{Frob}(F)$ as F with R-action given by $r \cdot v = \sqrt[p]{r} v$; this is only a heuristic, although it is literally true if R is perfect.)

Proposition 2.14. *When* $\mathrm{char}(R) = 2$*, for any finite free* R*-module* F *there is a natural short exact sequence*

$$0 \to \mathrm{Frob}(F) \xrightarrow{v \mapsto v \bullet v} \mathrm{S}^2(F) \xrightarrow{v \bullet w \mapsto v \wedge w} \Lambda^2(F) \to 0.$$

Proof. Pick a basis $x_1,\dots,x_n$ of F. Then this sequence is given by

$$0 \to R[x_1,\dots,x_n]_1 \xrightarrow{x_i\mapsto x_i^2} R[x_1,\dots,x_n]_2 \xrightarrow{\text{can}} \big(R[x_1,\dots,x_n]/(x_1^2,\dots,x_n^2)\big)_2 \to 0$$

which is readily verified to be exact. □

Remark 2.15. In particular, this shows that when $\operatorname{char}(R)=2$, Frob coincides with the second Adams operation; see below.

2.3. **Extending to operators on complexes: the "naive" approach.** We will want to extend the operations on free modules considered above to finite free complexes in a natural way and so that homotopy equivalence (quasi-isomorphism) is preserved. For simplicity, we'll focus on T^2, S^2 and Λ^2.

There is an evident candidate for T^2: Given a finite free complex $\mathbb{F}$, we may form the tensor product complex $\mathbb{F}\otimes_R\mathbb{F}$; see (1.2) above. Since $\mathbb{F}$ and $\mathbb{F}'$ are both finite free, if $\mathbb{F}\sim\mathbb{F}'$ then $\mathbb{F}\otimes_R\mathbb{F}\sim\mathbb{F}'\otimes_R\mathbb{F}\sim\mathbb{F}'\otimes_R\mathbb{F}'$, and so this meets our basic requirement.

Define an action of the symmetric group Σ_2 on $\mathbb{F}\otimes_R\mathbb{F}$ by

$$\tau(v\otimes w)=(-1)^{|v||w|}w\otimes v$$

where τ is the non-trivial element of Σ_2. The sign is present so that τ determines a chain map from $\mathbb{F}\otimes_R\mathbb{F}$ to itself. Define $\mathrm{S}^2_{\text{naive}}(\mathbb{F})$ to be the result of modding out by the S_2-action and $\Lambda^2_{\text{naive}}(\mathbb{F})$ to be the result of modding out by the signed S_2-action. That is

$$\mathrm{S}^2_{\text{naive}}(\mathbb{F})=R_{\text{trivial}}\otimes_{R[S_2]}(\mathbb{F}\otimes_R\mathbb{F})$$

and

$$\Lambda^2_{\text{naive}}(\mathbb{F})=R_{\text{sign}}\otimes_{R[S_2]}(\mathbb{F}\otimes_R\mathbb{F}).$$

Since τ respects the differential on $\mathbb{F}\otimes_R\mathbb{F}$, both of these are in fact complexes, with differential inherited from that on $\mathbb{F}\otimes_R\mathbb{F}$. In fact, using that there is an isomorphism of $R[S_2]$-modules

$$R[S_2]\cong R_{\text{trivial}}\oplus R_{\text{sign}} \tag{2.16}$$

(since 2 is assumed to be invertible), we have an isomorphism of chain complexes

$$\mathbb{F}\otimes_R\mathbb{F}\cong \mathrm{S}^2_{\text{naive}}(\mathbb{F})\oplus\Lambda^2_{\text{naive}}(\mathbb{F}). \tag{2.17}$$

Remark 2.18. Ignoring the differential and grading on $\mathbb{F}$, we have $\mathbb{F}=F_{\text{even}}\oplus F_{\text{odd}}$ as free R-modules, where $F_{\text{even}}=\bigoplus_i F_{2i}$ and $F_{\text{odd}}=\bigoplus_i F_{2i+1}$, and (still ignoring differentials and gradeds) we have $\mathrm{S}^2_{\text{naive}}(\mathbb{F})=\mathrm{S}^2(F_{\text{even}})\otimes_R\Lambda^2(F_{\text{odd}})$ and $\Lambda^2_{\text{naive}}(\mathbb{F})=\mathrm{S}^2(F_{\text{odd}})\otimes_R\Lambda^2(F_{\text{even}})$.

These naive operators behave well provided 2 is invertible:

Proposition 2.19. *Assume* 2 *has a multiplicative inverse in R. Then* S^2_{naive} *and* Λ^2_{naive} *preserve quasi-isomorphisms of finite free complexes. Moreover, if* $\mathbb{F}$ *has finite length homology then do both* $\mathrm{S}^2_{naive}(\mathbb{F})$ *and* $\Lambda^2_{naive}(\mathbb{F})$.

Proof. The first assertion holds since R_{trivial} and R_{sign} are projective $R[\Sigma_2]$-modules; see (2.16). The second assertion follows from the first and the fact that both constructions localize well: $\mathrm{S}^2_{R,\text{naive}}(\mathbb{F})_{\mathfrak{p}}\cong\mathrm{S}^2_{R_{\mathfrak{p}},\text{naive}}(\mathbb{F}_{\mathfrak{p}})$ and similarly for Λ^2_{naive}. □

Exercise 2.20. Let $\mathbb{F} = \mathrm{Kos}_R(x)$; that is, $\mathbb{F} = (0 \to R \cdot b \xrightarrow{\partial} R \cdot a \to 0)$ for formal symbols a and b of degrees 0 and 1, with $\partial(b) = xa$. Then

$$T^2(\mathbb{F}) = (0 \to F_2 \to F_1 \to F_0 \to 0)$$

with $F_2 = R \cdot (b \otimes b)$, $F_1 = R \cdot (a \otimes b) \oplus R \cdot (b \otimes a)$ and $F_0 = R \cdot (a \otimes a)$, with the action of τ given by $b \otimes b \mapsto -b \otimes b$, $a \otimes b \mapsto b \otimes a$, and $a \otimes a \mapsto a \otimes a$.

(1) Show that if 2 is invertible in R, then a basis of the free module underlying $\mathrm{S}^2_{\text{naive}}(\mathbb{F})$ is $\{\frac{1}{2}(a \otimes b + b \otimes a), a \otimes a\}$ and hence

$$\mathrm{S}^2_{\text{naive}}(\mathbb{F}) \cong (0 \to R \xrightarrow{x} R \to 0) = \mathrm{Kos}_R(x).$$

(2) Show that when 2 is a invertible, $\Lambda^2_{\text{naive}}(\mathbb{F})$ has basis $\{b \otimes b, a \otimes b - b \otimes a\}$, and thus

$$\Lambda^2_{\text{naive}}(\mathbb{F}) \cong (0 \to R \xrightarrow{x} R \to 0) \text{ with } R \text{ in degrees 1 and 2;}$$

i.e., $\Lambda^2(\mathbb{F}) \cong \Sigma^1 \mathrm{Kos}_R(x)$.

(3) Show that if $\mathrm{char}(R) = 2$, then

$$\mathrm{S}^2_{\text{naive}}(\mathbb{F}) \cong (0 \to R \xrightarrow{x} R \xrightarrow{2} R \to 0) = (0 \to R \xrightarrow{x} R \xrightarrow{0} R \to 0).$$

By taking $x = 1$, conclude that $\mathrm{S}^2_{\text{naive}}$ does not preserve homotopies in this case.

To construct well-behaved operators in characteristic 2, one needs to use the Dold-Kan Correspondence; see Remark 4.16 below. For now we stick to the case when $\mathrm{char}(k) \neq 2$.

2.4. **Adams operations.** Adams operations were introduced by J. F. Adams in 1962 to study vector bundles on spheres [Ada62]. We are interested in analogues of these defined on finite free complexes, as developed by Gillet and Soule [GS87]. In all contexts, Adams operations are endomorphisms of Grothendieck groups that are defined using formal sums of symmetric and exterior powers. I'll focus mostly on the *second Adams operation,* ψ^2, but I'll give some indication of he k-th Adams operation, ψ^k, for $k > 2$.

Definition 2.21. Assume $(R, \mathfrak{m}, k)$ is a local ring such that $\mathrm{char}(k) \neq 2$. The *second Adams operation*, ψ^2, sends a finite free complex $\mathbb{F}$ with finite length homology to the class

$$\psi^2(\mathbb{F}) = [\mathrm{S}^2_{\text{naive}}(\mathbb{F})] - [\Lambda^2_{\text{naive}}(\mathbb{F})] \in K_0^{fl}(R).$$

This is indeed a class in $K_0^{\mathrm{fl}}(R)$ by Proposition 2.19. More generally, the same formula determines a class in $K_0^Z(R)$ whenever $\mathbb{F}$ is a complex supported on Z.

Remark 2.22. See Remark 4.16 for a definition of ψ^2 that does not involve any restriction on the residue characteristic.

Recall S^2 and Λ^2 are polynomial functors of degree two on the category of finite free modules, and moreover their first cross effects functors coincide: $(\mathrm{S}^2)' \cong (\Lambda^2)'$ (both send a pair of free modules (F_1, F_2) to $F_1 \otimes_R F_2$). Because of this fact, it follows that

$$\begin{aligned}
\psi^2(\mathbb{F}_1 \oplus \mathbb{F}_2) &= [\mathrm{S}^2(\mathbb{F}_1 \oplus \mathbb{F}_2)] - [\Lambda^2(\mathbb{F}_1 \oplus \mathbb{F}_2)] \\
&= [\mathrm{S}^2(\mathbb{F}_1)] + [\mathrm{S}^2(\mathbb{F}_2)] + [\mathbb{F}_1 \otimes_R \mathbb{F}_2] - [\Lambda^2(\mathbb{F}_1)] - [\Lambda^2(\mathbb{F}_2)] - [\mathbb{F}_1 \otimes_R \mathbb{F}_2] \\
&= [\mathrm{S}^2(\mathbb{F}_1)] - [\Lambda^2(\mathbb{F}_1)] + [\mathrm{S}^2(\mathbb{F}_2)] - [\Lambda^2(\mathbb{F}_2)] \\
&= \psi^2(\mathbb{F}_1) + \psi^2(\mathbb{F}_2).
\end{aligned}$$

That is ψ^2 preserves addition of complexes. In fact, more is true:

Exercise 2.23. Show that if $0 \to \mathbb{F}' \to \mathbb{F} \to \mathbb{F}'' \to 0$ is a short exact sequence (not necessarily split) of finite free complexes, then

$$\psi^2(\mathbb{F}) = \psi^2(\mathbb{F}') + \psi^2(\mathbb{F}'').$$

Combining this with Proposition 2.19 proves the first assertion in the following result. (I will not prove the second assertion).

Proposition 2.24. *Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathrm{char}(k) \neq 2$. The function ψ^2 induces a group homomorphism*

$$\psi^2 : K_0^{\mathrm{f}l}(R) \to K_0^{\mathrm{f}l}(R).$$

More generally, for any closed subset Z of $\mathrm{Spec}(R)$, ψ^2 determines a group endomorphism of $K_0^Z(R)$ that is multiplicative in the following sense: Given another closed subset W, tensor product of complexes induces a pairing

$$K_0^Z(R) \otimes_{\mathbb{Z}} K_0^W(R) \xrightarrow{-\otimes-} K_0^{Z\cap W}(R)$$

and for any classes $\alpha \in K_0^Z(R)$ and $\beta \in K_0^W(R)$, we have

$$\psi^2(\alpha \otimes \beta) = \psi^2(\alpha) \otimes \psi^2(\beta).$$

Remark 2.25. In fact, both $\mathrm{S}^2_{\mathrm{naive}}$ and $\Lambda^2_{\mathrm{naive}}$ determine non-additive operators on $K_0^{\mathrm{f}l}(R)$.

Proposition 2.26. *Suppose $x_1, \dots, x_m \in \mathfrak{m}$ are such that $R/(x_1, \dots, x_m)$ has finite length and let $\mathbb{K} = \mathrm{Kos}_R(x_1, \dots, x_m)$. Then*

$$\psi^2(\mathbb{K}) = 2^m[\mathbb{K}].$$

Remark 2.27. $[\mathbb{K}] = 0$ if $m > \dim(R)$.

Proof. Recall $\mathrm{Kos}_R(x_1, \dots, x_m) = \otimes_j \mathrm{Kos}_R(x_i)$. Using (on faith) that ψ is multiplicative, we may reduce to the case of showing this for $m = 1$. In this case, by Exercise 2.20 we have

$$\psi^2(\mathrm{Kos}_R(x)) = [\mathrm{Kos}_R(x)] - [\Sigma \,\mathrm{Kos}_R(x)] = 2 \cdot [\mathrm{Kos}_R(x)].$$

□

Corollary 2.28. *If R is regular of Krull dimension d, then ψ^2 acts a multiplication by 2^d on $K_0^{\mathrm{f}l}(R)$. In particular, $\chi\Psi^2(\mathbb{F}) = 2^d\chi(\mathbb{F})$ for all finite free complexes $\mathbb{F}$ having finite length homology.*

Proof. This follows from the Proposition and Exercise 2.5. □

2.4.1. *A brief discussion of ψ^k for $k \geq 3$.* For $k \geq 2$, and provided $k!$ is a unit in R, one may define the k-th Adams operaration in terms of the "naive" exterior power operators $\Lambda^1_{\text{naive}}, \cdots, \Lambda^k_{\text{naive}}$. When $k!$ is not invertible, the Dold-Kan correspondence needs to be used; see Section 4.1 below.

In detail, when $k!$ is a unit, we set

$$\psi^k(\mathbb{F}) = N_k(\Lambda^1_{\text{naive}}(\mathbb{F}), \dots, \Lambda^k_{\text{naive}}(\mathbb{F}))$$

where N_k is the k-th Newton polynomial, defined by the condition that

$$N_k(s_1, \dots, s_k) = \sum_i x_i^k$$

where $s_1, \dots, s_k$ are the elementary symmetric polynomials in $x_1, \dots, x_k$.

For instance when $k = 2$, $N_2(x, y) = x^2 - 2y$ and so

$$\psi^2(\mathbb{F}) = [\mathbb{F} \otimes_R \mathbb{F}] - 2[\Lambda^2_{\text{naive}}(\mathbb{F})] = [\mathrm{S}^2_{\text{naive}}(\mathbb{F})] - [\Lambda^2_{\text{naive}}(\mathbb{F})]$$

with the second equality holding by (2.17).

When $k = 3$, we have $N_3(x, y, z) = x^3 - 3xy + 3z$ and so

$$\psi^3(\mathbb{F}) = [\mathbb{F} \otimes_R \mathbb{F} \otimes_R \mathbb{F}] - 3[\mathbb{F} \otimes \Lambda^2_{\text{naive}}(\mathbb{F})] + 3[\Lambda^3_{\text{naive}}(\mathbb{F})].$$

The operators ψ^k enjoy the same formal properties as does ψ^2. For our purposes ψ^2 turns out to be of much greater value and so we focus on it from now on.

3. Proofs of some of the algebraic rank conjectures

In this section we give proofs of some of the algebraic rank conjectures.

3.1. **Proof for quasi-Roberts rings.** Assume $(R, \mathfrak{m}, k)$ is a local ring of dimension d with $\text{char}(k) \neq 2$. We say R is *quasi-Roberts ring* if there is an equality $\chi \circ \psi^2 = 2^d \cdot \psi^2$ of operators on $K_0^{fl}(R)$; that is, if

$$\chi(\mathrm{S}^2_{\text{naive}}(\mathbb{F})) - \chi(\Lambda^2_{\text{naive}}(\mathbb{F})) = 2^d \cdot \chi(\mathbb{F})$$

for any finite free complex $\mathbb{F}$ having finite length homology. (The same definition applies even if $\text{char}(k) = 2$, but in that case we will need to use the "non-naive" symmetric and exterior powers; see Remark 4.16 below.)

By Corollary 2.28, regular local rings are quasi-Roberts rings. So are *complete intersection rings* — i.e., rings of the form $R = Q/(f_1, \cdots, f_c)$ with Q a regular local ring and $f_1, \dots, f_c$ a regular sequence of elements, and a smattering of other examples; see [Kur01].

Theorem 3.1. [Wal17] *Assume $(R, \mathfrak{m}, k)$ is a quasi-Roberts ring — e.g., regular or a complete intersection — such that $\text{char}(k) \neq 2$, and let d be its Krull dimension. For any finite free complex $\mathbb{F}$ of R-modules with finite length homology, we have*

$$\text{rank}(\mathbb{F}) \cdot h(\mathbb{F}) \geq 2^d \cdot |\chi(\mathbb{F})| \tag{3.2}$$

In particular, if M is a non-zero R-module of finite length and finite projective dimension then $\text{rank}(\mathbb{F}) \geq 2^{\dim(R)}$ for any free resolution $\mathbb{F}$ of M.

Proof. The latter assertion follows from (3.2) since $h(\mathbb{F}) = \chi(\mathbb{F}) = \text{length}_R(M) \neq 0$ in that case.

Let $\mathbb{F}$ be any finite free complex with finite length homology. We set-up/recall some notation:

- $h_{\text{even}}(\mathbb{F}) = \sum_j \text{length}_R H_{2j}(\mathbb{F})$,

- $h_{\mathrm{odd}}(\mathbb{F}) = \sum_j \operatorname{length}_R H_{2j+1}(\mathbb{F})$,
- $h(\mathbb{F}) = h_{\mathrm{even}}(\mathbb{F}) + h_{\mathrm{odd}}(\mathbb{F}) = \sum_i \operatorname{length}_R H_i(\mathbb{F})$, and
- $\chi(\mathbb{F}) = h_{\mathrm{even}}(\mathbb{F}) - h_{\mathrm{odd}}(\mathbb{F}) = \sum_i (-1)^i \operatorname{length} H_i(\mathbb{F})$.

The Theorem follows from the pair of Key Inequalities

$$\operatorname{rank}(\mathbb{F}) \cdot h(\mathbb{F}) \geq h(\mathbb{F} \otimes_R \mathbb{F}) \geq 2^d \cdot \chi(\mathbb{F}), \tag{3.3}$$

since we may assume that $\chi(\mathbb{F}) \geq 0$ by replacing $\mathbb{F}$ with $\Sigma\mathbb{F}$ if necessary.

Since $\operatorname{char}(k) \neq 2$, 2 is invertible in R and hence from (2.17) we have a natural isomorphism

$$\mathbb{F} \otimes_R \mathbb{F} \cong \Lambda^2_{\mathrm{naive}}(\mathbb{F}) \oplus \mathrm{S}^2_{\mathrm{naive}}(\mathbb{F}).$$

This gives the first equality in

$$\begin{aligned} h(\mathbb{F} \otimes_R \mathbb{F}) &= h(\mathrm{S}^2_{\mathrm{naive}}(\mathbb{F})) + h(\Lambda^2_{\mathrm{naive}}(\mathbb{F})) \\ &\geq h_{\mathrm{even}}(\mathrm{S}^2_{\mathrm{naive}}(\mathbb{F})) + h_{\mathrm{odd}}(\Lambda^2_{\mathrm{naive}}(\mathbb{F})) \\ &\geq \chi(\mathrm{S}^2_{\mathrm{naive}}(\mathbb{F})) - \chi(\Lambda^2_{\mathrm{naive}}(\mathbb{F})) \\ &= \chi\psi^2(\mathbb{F}) \\ &= 2^d \cdot \chi(\mathbb{F}) \end{aligned}$$

The inequalities are elementary and the final two equalities hold by definition of ψ^2 and the fact that we assume R is quasi-Roberts. This establishes the right-hand inequality of (3.3).

The left-hand inequality in (3.3) is elementary, but I will state it as a lemma below, since it is needed again later. In detail, the proof of the Theorem is complete by applying Lemma 3.5 with $\mathbb{M} = \mathbb{F}$. □

Remark 3.4. By analyins the proof it is not hard to see that if M is a module of finite length and finite projective dimension such that and $\operatorname{rank}(\mathbb{F}) = 2^d$ for some resolution $\mathbb{F}$, then M must be cyclic, i.e., $M \cong R/I$, and $I/I^2 \cong \operatorname{Tor}_1^R(R/I, R/I)$ is free as an R/I-module. The latter implies I is generated by a regular sequence thanks to a theorem of Ferrand and Vasconcelos; see [BH98, 2.2.8]. Thus, equality in the Total Rank Conjecture holds for a module M if and only if M is isomorphic to the ring modulo a regular sequence of parameters.

I owe you:

Lemma 3.5. *For any commutative ring R, any finite free complex $\mathbb{F}$, and any bounded complex $\mathbb{M}$ of R-modules having finite length homology, we have*

$$h(\mathbb{F} \otimes_E \mathbb{M}) \leq \operatorname{rank}(\mathbb{F}) h(\mathbb{M}). \tag{3.6}$$

Exercise 3.7. Prove Lemma 3.5 using one of the following two approaches:

(1) Exploit the spectral sequence

$$H_i(\mathbb{F} \otimes_R H_j\mathbb{M}) \Rightarrow H_{i+j}(\mathbb{F} \otimes_R \mathbb{M})$$

using also that $h(\mathbb{F} \otimes N) \leq \operatorname{rank}(\mathbb{F}) \operatorname{length}(N)$ for any module N having finite length.

(2) Alternatively, proceed by induction on $h(\mathbb{M})$. For the case $h(\mathbb{M})$ use that we may assume $\mathbb{M} = k$.

3.2. **Extension to dg S-modules.** Recall $S = k[x_1, \dots, x_d]$ for a field k and with $\deg(x_i) = -2$.

Theorem 3.8. [Wal17] *Assume* $\mathrm{char}(k) \neq 2$. *For any semi-free dg S-module $\mathbb{F}$ having finite length homology, we have*

$$\mathrm{rank}(\mathbb{F})h(\mathbb{F}) \geq 2^d \cdot |\chi(\mathbb{F})|.$$

Moreover, the Total Rank Conjecture for Formal dg Modules holds.

Sketch of Proof. The proof of the first assertion is essentially identical to the proof of the analogous fact in the local quasi-Roberts ring case.

Since S is a regular ring, the Total Rank Conjecture for Formal dg Modules is equivalent to the statement that if M is a graded S-module of finite length, then $\mathrm{rank}(\mathbb{F}) \geq 2^d$ for the minimal, semi-free dg resolution $\mathbb{F}$ of M (regarded as a dg-S-module with trivial differential). Since S is concentrated in even degrees, one easily reduces to the case when M is either concentrated in even degrees or concentrated in odd degrees. In this case $h(\mathbb{F}) = h(M) = |\chi(M)| = |\chi(\mathbb{F})|$. $\square$

Exercise 3.9. Complete the details of this proof by making suitable modifications to the proof of Theorem 3.1.

3.3. **Proof in characteristic $p \geq 3$.** Recall the bilinear pairing

$$\chi(-,-) : K_0^{\mathrm{fl}}(R) \otimes_{\mathbb{Z}} G_0(R) \to \mathbb{Z}.$$

determined by $\chi(\mathbb{F}, M) = \chi(\mathbb{F} \otimes_R M)$. It will be convenient to tensor $\mathbb{Q}$ to make this a bilinear pairing of $\mathbb{Q}$-vector spaces:

$$\chi(-,-) : K_0^{\mathrm{fl}}(R)_{\mathbb{Q}} \otimes_{\mathbb{Q}} G_0(R)_{\mathbb{Q}} \to \mathbb{Q}$$

where $K_0^{\mathrm{fl}}(R)_{\mathbb{Q}} := K_0^{fl}(R) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $G_0(R)_{\mathbb{Q}} := G_0(R) \otimes_{\mathbb{Z}} \mathbb{Q}$.

A key property is the following:

Theorem 3.10. *Assume $(R, \mathfrak{m}, k)$ is a local ring of dimension d that is isomorphic to the quotient of a regular local ring. (The latter holds, for instance, if R is complete.) Then there is an internal direct sum decomposition of the rational vector space $G_0(R)_{\mathbb{Q}}$ of the form*

$$G_0(R)_{\mathbb{Q}} = \bigoplus_{i=0}^{\dim(R)} G_0(R)_{(i)} \tag{3.11}$$

with the following properties:

1. *For any class $\alpha \in K_0(R)_{\mathbb{Q}}$ and $\beta \in G_0(R)_{(i)}$ we have*
$$\chi(\psi^2\alpha, \beta) = 2^i \cdot \chi(\alpha, \beta).$$
(More gennerally, $\chi(\psi^k\alpha, \beta) = k^i\chi(\alpha, \beta)$.)
2. *For any m, the subspace $\bigoplus_{i=0}^m G_0(R)_{(i)}$ coincides with the subspace of $G_0(R)_{\mathbb{Q}}$ generated by classes of finitely generated modules M such that $\dim(M) \leq m$.*
3. *When $\mathrm{char}(R) = p$, R is complete, and k is a perfect field, we have $G_0(R)_{(i)} = \ker(\phi_* - p^i)$ where ϕ_* is the operator on $G_0(R)_{\mathbb{Q}}$ induced by restriction of scalars along the Frobenius map $\phi : R \xrightarrow{r \mapsto r^p} R$. (This map is module-finite by out assumptions on R.)*

Proof. I just give the basic idea: If $R = Q/I$ with Q regular local, then the map sending a finitely generated R-module M to a chosen free resolution over Q induces an isomorphism

$$G_0(R) \cong K_0^{V(I)}(Q), \tag{3.12}$$

with the target being the Grothendieck group of finite free complexes $\mathbb{F}$ supported on $V(I) = \{\mathfrak{q} \in \operatorname{Spec}(R) \mid \mathfrak{q} \supseteq I\}$. Now, $K_0^{V(I)}(R)$ admits Adams operations, ψ^k for $k \geq 2$, defined in the same way as for $K_0^{\mathrm{f}l}(R) = K_0^{\{\mathfrak{m}\}}(R)$ — see [GS87]. Moreover, as proven in [GS87], since Q is regular, for any $k \geq 2$, we get a decomposition

$$K_0^{V(I)}(Q)_{\mathbb{Q}} = \bigoplus_{j=0}^{c} K_0^{V(I)}(Q)^{(j)}$$

where $c = \dim(Q) - \dim(R)$ and $K_0^{V(I)}(Q)^{(j)} = \ker(\psi^k - k^j)$, the eigenspace of ψ^k of eigenvalue k^j. This determines the decomposition on $G_0(R)_{\mathbb{Q}}$ — specifically, we set $G_0(R)_{(i)}$ to be the subspace corresponding to $K_0^{V(I)}(R)^{(\dim(Q)-i)}$ under (3.12). The desired properties hold because the pairing $-\widetilde{\cap}- : K_0^{\mathrm{f}l}(R)_{\mathbb{Q}} \otimes_{\mathbb{Q}} K_0^{V(I)}(Q)_{\mathbb{Q}} \to \mathbb{Q}$ determined by (3.12) and $\chi(-,-)$ satisfies $\psi^k(\alpha)\widetilde{\cap}\psi^k(\beta) = k^{\dim(Q)}(\alpha\widetilde{\cap}\beta)$; see [Wal21, Lemma 4.5]. $\square$

Remark 3.13. There is an isomorphism $\tau : G_0(R)_{(i)} \xrightarrow{\cong} CH_i(R) \otimes_{\mathbb{Z}} \mathbb{Q}$, the rationalized Chow group of dimension i cycles on $\operatorname{Spec}(R)$.

Given $\beta \in G_0(R)$, there is a unique decomposition $\beta = \sum_i \beta_{(i)}$ with $\beta_{(i)} \in G_0(R)_{(i)}$, and we have $\chi(\psi^2\alpha, \beta) = \sum_i 2^i \chi(\alpha, \beta_{(i)})$. In particular, $\beta = \beta_{(j)}$ for some j (i.e., $\beta_{(i)} = 0$ for all $i \neq j$) if and only if $\chi(\psi^2\alpha, \beta) = 2^j\chi(\alpha, \beta)$ In general it is not easy to calculate the components $\beta_{(i)}$ of a class β, but there is one exception:

Lemma 3.14. *For a domain R, the map taking a finitely generated R-module to its rank induces an isomorphism $G_0(R)_{(d)} \xrightarrow{\cong} \mathbb{Q}$.*

Exercise 3.15. Prove Lemma 3.14 using the second property in Theorem 3.10 and the following "localization" exact sequence for G-theory: the sequence

$$\bigoplus_{f \neq 0} G_0(R/f) \to G_0(R) \to G_0(F) \to 0$$

(with the maps being the canonical ones) is exact, where F is the field of factions of R.

Example 3.16. When R is regular local, we have $G_0(R) \cong K_0(R) \cong \mathbb{Z}$, generated by the class $[R]$. In this case $G_0(R)_{(i)} = 0$ unless $i = d$, in which case it is all of $G_0(R)_{\mathbb{Q}}$. In particular, $[R] \in G_0(R)_{(d)}$. Since $\chi(\mathbb{F}, R) = \chi(\mathbb{F})$, we have

$$\chi(\psi^2\mathbb{F}) = 2^d \cdot \chi(\mathbb{F}).$$

That is, R is quasi-Roberts, as noted before.

This formula holds, more generally, for a local ring R for which $[R]$ happens to be equal to $[R]_{(d)}$ (or, equivalently, $[R]_{(i)} = 0$ for all $i \neq d$). Such rings are known as *Roberts rings* — see [Kur01] for examples. (Every Roberts ring is quasi-Roberts, but the converse can fail since $\chi(\alpha, [R]_{(i)})$ can be 0 for all α even if $[R]_{(i)} \neq 0$.)

Theorem 3.17. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d such that* $\operatorname{char}(k) \neq 2$. *Suppose there is a non-zero R-module M with the following properties*

(1) *M is Maximum Cohen-Macaulay (MCM) — this means there exists a sequence $x_1, \dots, x_d \in \mathfrak{m}$ that is M-regular; i.e., $H_i(\mathrm{Kos}_R(x_1, \dots, x_d) \otimes_R M) = 0$ for $i \geq 1$.*

(2) *$[M] \in G_0(R)_{\mathbb{Q}}$ is pure of weight d.*

Then the Total Rank Conjectures 1.12 holds for R.

Proof. The proof may be viewed as a generalization of the proof of Theorem 3.1; by taking $R = M$ in this proof, we will recover that proof.

Let $\mathbb{F}$ be an arbitrary finte free complex with finite length homology. The Key Inequalities for this proof are

$$\operatorname{rank}(\mathbb{F}) \cdot h(\mathbb{F} \otimes_R M) \geq h(\mathbb{F} \otimes_R \mathbb{F} \otimes_R M) \geq 2^d \cdot \chi(\mathbb{F} \otimes_R M). \tag{3.18}$$

The left inequality follows from Lemma 3.5 above (and it uses none of the assumption on M). The right one follows from

$$\begin{aligned} h(\mathbb{F} \otimes_R \mathbb{F} \otimes_R M) &= h(\mathrm{S}^2_{\mathrm{naive}}(\mathbb{F}) \otimes_R M) + h(\Lambda^2_{\mathrm{naive}}(\mathbb{F}) \otimes_R M) \\ &\geq h_{\mathrm{even}}(\mathrm{S}^2_{\mathrm{naive}}(\mathbb{F}) \otimes_R M) + h_{\mathrm{odd}}(\Lambda^2_{\mathrm{naive}}(\mathbb{F}) \otimes_R M) \\ &\geq \chi(\mathrm{S}^2_{\mathrm{naive}}(\mathbb{F}) \otimes_R M) - \chi(\Lambda^2_{\mathrm{naive}}(\mathbb{F}) \otimes_R M) \\ &= \chi(\psi^2(\mathbb{F}), M) \\ &= 2^d \cdot \chi(\mathbb{F}, M) \end{aligned}$$

which is justified in exactly the same was as in the proof of Theorem 3.1 — the last inequality uses that $[M]$ is pure of weight d by assumption.

Now assume $\mathbb{F}$ is a non-trivial tiny complex. Since M is MCM and $\mathbb{F}$ is tiny, a standard argument (see the exercise below) shows that $\mathbb{F} \otimes_R M$ only has homology in degree 0; i.e., $H_i(\mathbb{F} \otimes_R M) = 0$ for all $i \neq 0$. Thus

$$\chi(\mathbb{F} \otimes_R M) = h(\mathbb{F} \otimes_R M) = \operatorname{length} H_0(\mathbb{F} \otimes_R M) = \operatorname{length}(H_0(\mathbb{F}) \otimes_R M) \neq 0.$$

and so the result follows from (3.18). □

Exercise 3.19. Show that if $\mathbb{F}$ is tiny and M is an MCM module, then $H_i(\mathbb{F} \otimes_R M) = 0$ for all $i \neq 0$. *Tip*: Consider the bicomplex $\mathbb{F} \otimes_R \mathcal{C} \otimes_R M$ where $\mathcal{C}$ is the "algebraist's" Cech complex on a system of parameters $x_1, \dots, x_d \in \mathfrak{m}$ of R:

$$\mathcal{C} = (0 \to R \to \bigoplus_i R\left[\frac{1}{x_i}\right] \to \bigoplus_{i<j} R\left[\frac{1}{x_i x_j}\right] \to \cdots \to R\left[\frac{1}{x_1 \cdots x_d}\right] \to 0).$$

You may use that M is MCM if and only if $\mathcal{C} \otimes_R M$ has homology only in the right-most position. (The homology of $\mathcal{C} \otimes_R M$ gives $H^*_{\mathfrak{m}}(M)$, the local cohomology of M supported at the maximal ideal.)

Unfortunately, I know of few examples, outside of the case when R is a CM quasi-Roberts ring, when such an M as in Theorem 3.17 exists. However, an analysis of the proof of this theorem reveals that we just need a sequence of modules $M_1, M_2, \cdots$ that "asymptotically" satisfy these properties. Let me explain.

To simplify the exposition, let us assume that R is an integral domain for the rest of this section (but that is not needed).

Definition 3.20. Assume $(R, \mathfrak{m}, k)$ is local domain of dimension d. Let $\{M_j\} = M_1, M_2, \dots$ be a sequence of finitely generated R-modules such that $\operatorname{rank}_R(M_j) > 0$ for all j.

- We say $\{M_i\}$ is *lim MCM* if
$$\lim_{j\to\infty} \frac{\operatorname{length} H_i \operatorname{Kos}_{M_j}(x_1, \dots, x_d)}{\operatorname{rank}_R(M_j)} = 0 \text{ for all } i > 0$$
for some (equivalently, any) system of parameters $x_1, \dots, x_d$ of R.
- Such a sequence is *lim of pure weight* d if
$$\lim_{j\to\infty} \frac{[M_j]_{(i)}}{\operatorname{rank}_R(M_j)} = 0 \text{ for all } 0 \le i < d.$$
Here, for a finitely generated module M, we write $[M]_{(i)}$ denote the component of $[M] \in G_0(R)_{\mathbb{Q}}$ in the summand $G_0(R)_{(i)}$ under the internal direct sum decomposition given in Theorem 3.10. The limit here should be interpreted as occurring with the finite subspace topology on the (possibly infinite dimensional) $\mathbb{Q}$-vector space $G_0(R)_{(i)}$.

Remark 3.21. When R is a domain, $[M]_{(d)} = \operatorname{rank}(M)[R]_{(d)}$ and so, if we allowed $i = d$ in the definition of lim of pure weight d, the limit would be $[R]_{(d)}$.

The main example of sequences satisfying both definitions is the following:

Example 3.22. Suppose $\operatorname{char}(R) = p > 0$ and assume R is a complete domain with perfect residue field. Let M_j denote R regarded as an R-module via restriction of scalars along the j-th iterate, $\phi^j : R \xrightarrow{r \mapsto r^{p^j}} R$, of the Frobenius endomorphism. Then $\{M_j\}$ is lim CM by [Hoc17, Theorem 5.1].

It is also lim of pure weight d. To show this, we use that ϕ^j_* acts on $G_0(R)_{(i)} \subseteq G_0(R)_{\mathbb{Q}}$ as multiplication by 2^{ij}; see Theorem 3.10 (3). We have $[R] = \sum_i [R]_{(i)}$ and so
$$[M_j] = \phi^j_*[R] = \sum_i 2^{ji}[R]_{(i)}.$$
which gives
$$[M_j]_{(i)} = 2^{ji}[R]_{(i)}.$$
In particular, $\operatorname{rank}(M_j) = 2^{dj}$. So, for $0 \le i < d$ we have
$$\lim_{j\to\infty} \frac{[M_j]_{(i)}}{\operatorname{rank}_R(M_j)} = \lim_{j\to\infty} \frac{2^{ji}}{2^{dj}}[R]_{(i)} = 0 \cdot [R]_{(i)} = 0.$$

Theorem 3.23. *Assume $(R, \mathfrak{m}, k)$ is a local domain and* $\operatorname{char}(k) \neq 2$. *If R admits a sequence $\{M_j\}$ that is both lim MCM and lim of pure weight $d = \dim(R)$, then the Total Rank Conjecture 1.12 holds for R.*

Proof. The proof is basically the same as the proof of Theorem 3.17, using limits as needed. □

Corollary 3.24. *The Total Rank Conjecture holds for any local ring of characteristic $p \ge 3$.*

Proof. A standard argument allows us to reduce to the case when R is complete with perfect residue field. By modding out by a minimal prime $\mathfrak{p}$ such that $\dim(R/\mathfrak{p}) = \dim(R)$, we may assume R is a domain. (Note that even if we start with $\mathbb{F}$ being the resolution of a module, $\mathbb{F}/\mathfrak{p}\mathbb{F}$ might not be a resolution, but it remains tiny.) Then the result follows from the theorem and Example 3.22. □

3.3.1. *Extension to char 0.* Theorem 3.1 and 3.17 give that the Total Rank Conjecture holds for any local ring that is an algebra over a field k of positive characteristic. We also have:

Theorem 3.25 (VandeBogert-W)**.** [VW24] *The Total Rank Conjecture holds for any local ring that is an algebra over a field of characteristic* 0*.*

Sketch of Proof. A well-worn technique, sometimes known as "Hochster's Meta-Theorem" [Hoc75] allows one to reduce to the prime characteristic case. I only give the vague idea: Suppose R is an algebra over a field of characteristic 0 and $\mathbb{F}$ is a hypothetical counter-example ot the Total Rank Conjeture. Then, for all but a finite number of prime intergers p, one may construct a ring R_p and finite free R_p-complex $\mathbb{F}_p$ such that $\mathrm{char}(R_p) = p$ and "all relevant properties" are preserved: $\dim(R_p) = \dim(R)$, $\mathrm{length}_{R_p} H_i(\mathbb{F}_p) = \mathrm{length}_R H_i(\mathbb{F})$ for all i, and $\mathrm{rank}_i^{R_p}(\mathbb{F}_p) = \mathrm{rank}_i^R(\mathbb{F})$ for all i. Since the TRC holds for R_p, this is not possible. □

3.4. **Counter-examples to the TRC for Complexes.** Recall that the Total Rank Conjecture for Complexes predicts that $\beta(\mathbb{F}) \geq 2^d$ whenever $\mathbb{F}$ is a (not necessarily tiny) finite free complex with finite length, non-zero homology. We present counter-examples, discovered by Iyengar and myself [IW18], that arise even in the very nice situation in which R is a regular local ring. All our counter examples do require that the residue characteristic is not 2, however.

For simplicity, let us take $R = k[[x_1, \dots, x_d]]$ with k a field of characteristic $\neq 2$. Recall from Theorem 3.1 that

$$\mathrm{rank}(\mathbb{F}) \geq 2^d \cdot \frac{|\chi(\mathbb{F})|}{h(\mathbb{F})}.$$

Thus, an obvious place to look for a counter example to the Total Rank Conjecture for Complexes is complexes with $\chi(\mathbb{F}) = 0$, and these are easy to come by:

Start with the Koszul complex $\mathbb{K} = \mathrm{Kos}_R(x_1, \dots, x_d)$, which resolves k in this case, and pick an endomorphism g of $\mathbb{K}$ of degree -2. In other words, take a chain map $g : \Sigma^{-1}\mathbb{K} \to \Sigma\mathbb{K}$. Now set $\mathbb{F} = \mathrm{cone}(g)$. This gives a short exact sequence

$$0 \to \Sigma\mathbb{K} \to \mathbb{F} \to \mathbb{K} \to 0$$

of complexes with finite length homology. It follows that $[\mathbb{F}] = 0$ in $K^{fl}(R)$, whence $\chi(\mathbb{F}) = 0$.

To calculate the Betti numbers of $\mathbb{F}$, we use that

$$0 \to \mathrm{Hom}_R(\mathbb{K}, k) \to \mathrm{Hom}_R(\mathbb{F}, k) \to \Sigma^{-1} \mathrm{Hom}_R(\mathbb{K}, k) \to 0$$

is exact and $\mathrm{Hom}_R(\mathbb{K}, k) \sim \Lambda_k(e_1^*, \dots, e_d^*)$ with e_i^* of cohomological degree 1. This gives the long exact sequence

(3.26)
$$\cdots \to \Lambda^{j-1}(e_1^*, \dots, e_d^*) \to \Lambda^{j+1}(e_1^*, \dots, e_d^*) \to H^j \mathrm{Hom}_R(\mathbb{F}, k) \to \Lambda^j(e_1^*, \dots, e_d^*) \to \Lambda^{j+2}(e_1^*, \dots, e_d^*) \to \cdots.$$

In particular, the ranks of the maps $\Lambda^*(e_1^*, \dots, e_d^*) \to \Lambda^{*+2}(e_1^*, \dots, e_d^*)$ determine the Betti numbers $\beta_i(\mathbb{F}) = \dim_k \mathrm{Hom}_R^i(\mathbb{F}, k)$ of $\mathbb{F}$. (Recall $\beta(\mathbb{F}) = \mathrm{rank}(\overline{\mathbb{F}})$ where $\overline{\mathbb{F}}$ is a minimal complex homotopy equivalent to $\mathbb{F}$.)

Now, the homotopy class of such a g is represented by an element

$$q \in H^2(\mathrm{End}_R(\mathbb{F})) \cong H^2(\mathrm{Hom}_R(\mathbb{F}, k)) = \Lambda^2(e_1^*, \dots, e_d^*).$$

and it is not difficult to see that the map $\Lambda^*(e_1^*, \ldots, e_d^*) \to \Lambda^{*+2}(e_1^*, \ldots, e_d^*)$ occurring in (3.26) is multiplication by q. Specifically we take

$$q = e_1^* e_2^* + e_3^* e_4^* + \cdots + e_{d-1}^* e_d^*.$$

(Recall we assume d is even.)

Example 3.27. A toy example: Take $d = 2$, so that g is the map associated to $q = e_1^* e_2^*$. Then $\mathbb{F}$ is the totalization of the complex

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R^1 \arrow[r] \arrow[d, "="] & R^2 \arrow[r] \arrow[d] & R^1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & R^1 \arrow[r] & R^2 \arrow[r] & R^1 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

Note it is not minimal, and its minimization has the form

$$\overline{\mathbb{F}} = (0 \to R^1 \to R^2 \to R^2 \to R^1 \to 0).$$

Example 3.28. For $d = 4$, $\mathbb{F}$ is the totalization of

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R^1 \arrow[r] \arrow[d, hook] & R^4 \arrow[r] \arrow[d, "="] & R^6 \arrow[r] \arrow[d, two heads] & R^4 \arrow[r] \arrow[d] & R^1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & R^1 \arrow[r] & R^4 \arrow[r] & R^6 \arrow[r] & R^4 \arrow[r] & R^1 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

and its minimization is

$$\overline{\mathbb{F}} = (0 \to R^1 \to R^4 \to R^5 \to R^5 \to R^4 \to R^1 \to 0).$$

Lemma 3.29. [IW18, 2.1] *With the notation above, assume d is even and set $q = \sum_{i=1}^{d/2} e_{2i-1}^* e_{2i}^*$. Provided $\operatorname{char}(k) = 0$ or $\operatorname{char}(k) > \frac{d}{4} + \frac{1}{2}$ the map*

$$q : \Lambda^j(e_1^*, \ldots, e_d^*) \to \Lambda^{j+2}(e_1^*, \ldots, e_d^*)$$

has full rank — i.e., it is injective for $j \leq d/2 - 1$ and surjective for $j \geq d/2 - 1$.

Consequently, we have:

Proposition 3.30. [IW18] *The minimal complex $\overline{\mathbb{F}}$ constructed above has the following properties:*

- *It is a non-trivial finite free complex — in fact, $H_i(\mathbb{F}) = k$ for $i = 0$ or $i = 1$ and $H_i(\mathbb{F}) = 0$ for all other values of i.*
- *$\mathbb{F}$ is concentrated in degrees $[0, d+1]$.*
- $\operatorname{rank}(\overline{\mathbb{F}}) = \binom{d+2}{\frac{d}{2}+1}$.

In particular, when $d \geq 8$ and $\operatorname{char}(k) = 0$ or $\operatorname{char}(k) > \frac{d}{4} + \frac{1}{2}$, the Total Rank Conjecture for Complexes is false for $R = k[[x_1, \ldots, x_d]]$.

Exercise 3.31. Deduce Proposition 3.30 from Lemma 3.29.

Very similar constructions give the next two results.

Proposition 3.32. [IW18] *The Total Rank Conjecture for dg Modules is false.*

These counter-examples do not imply the To*r*al Rank Conjecture is false, however, since the examples cannot possibly "come from topology" as they are not dgas.

Proposition 3.33. [IW18] *The Algebraic Version of Carlsson's Conjecture 1.24 is false when* $\mathrm{char}(k) \neq 2$ *and* $n \geq 8$.

Exercise 3.34. Prove Proposition 3.33 in the case when $n \geq 8$ is even, as follows: Recall $R = k[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$ with $\mathrm{char}(k) = p$. Let $\mathbb{K} = \mathrm{Kos}_R(x_1, \dots, x_n)$.

(1) Show $H_*(\mathbb{K})$ is an exterior algebra over k generated by $H_1(\mathbb{K}) \cong k^n$.
(2) Let $z \in K_2$ be a cycle representing the class of an element $q \in H_2(\mathbb{K})$ as in Lemma 3.29, and set $\mathbb{F}$ to be the cone of multiplication by z. Show $\mathbb{F}$ is a counter-example by applying Lemma 3.29.

As with the previous counter-examples, these do not show the original Carlsson's Conjecture 1.23 is false — Rüping and Stephan [RS22] have shown that these counter-examples do not come from the action of a p-torus on a space.

4. Characteristic two

In this section we discuss what is known about the various Rank Conjectures when the characteristic of the ring is 2. This is the most interesting case, since none of the proofs given so far, and none of the counter-examples presented so far, have anything to say about the characteristic 2 case. Assume henceforth that $(R, \mathfrak{m}, k)$ is a local ring with $\mathrm{char}(R) = 2$. An important ingredient in this case will be the Dold-Kan Correspondence.

4.1. Dold-Kan Correspondence.

For $n \in \mathbb{Z}_{\geq 0}$, set $[n]$ to be the totally ordered finite non-empty set
$$[n] := \{0, 1, \dots, n\}.$$
Define Ord to be the category with objects $[n]$, for $n \geq 0$, and morphisms given by order preserving functions:
$$\mathrm{Hom}_{\mathrm{Ord}}([n], [m]) = \{f : [n] \to [m] \mid i \leq j \Rightarrow f(i) \leq f(j)\}.$$
A *simplicial object in a category* $\mathcal{C}$ is a contravariant functor $X : \mathrm{Ord} \to \mathcal{C}$.

In concrete terms, a simplicial object X is a sequence of objects $X_0, X_1, \dots$ of $\mathcal{C}$ together with *face* morphism $d_j : X_{n+1} \to X_n$, for $0 \leq j \leq n$, corresponding to the unique injection $\delta_j : [n] \hookrightarrow [n+1]$ that "skips j" and *degeneracy* morphism $s_j : X_n \to X_{n+1}$, for $0 \leq j \leq n$, is corresponding to the unique surjection $\sigma_j : [n+1] \twoheadrightarrow [n]$ that "repeats j". These face and degeneracy maps satisfy the following simplicial identities:
$$d_i d_j = d_{j-1} d_i \text{ if } i < j$$
$$s_i s_j = s_j s_{i-1} \text{ if } i > j$$
$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases}$$
Simplicial objects in $\mathcal{C}$ form a category in which a morphism is given by a natural transformation of functors from $\mathrm{Ord}^{\mathrm{op}}$ to $\mathcal{C}$ or, equivalently, by a sequence of morphisms $f_n : X_n \to Y_n, n \geq 0$ in $\mathcal{C}$ that commute with the face and degeneracy maps for simplicial objects $X_\bullet$ and $Y_\bullet$.

A *simplicial homotopy* joining a pair of simplicial maps $f, g : X_\bullet \to Y_\bullet$ between two simplicial objects $X_\bullet$ and $Y_\bullet$ is a collection of morphisms $h_n : X_n \to Y_{n+1}$ for $n \geq 0$ such that the following identities hold:

$$\begin{aligned} d_0 h_n &= f_n \text{ for all } n \geq 0 \\ d_{n+1} h_n &= g_n \text{ for all } n \geq 0 \\ d_i h_j &= \begin{cases} h_{j-1} d_i, & i < j \\ d_i h_{i-1}, & i = j \neq 0 \\ h_j d_{i-1}, & i > j+1 \end{cases} \\ s_i h_j &= \begin{cases} h_{j+1} s_i, & i \leq j \\ h_j s_{i-1}, & i > j \end{cases} \end{aligned} \tag{4.1}$$

The details of these equations do not matter, except to note that a simiplicial homotopy involves only equalities of compositions of functions.

Let us now focus on the case of simplicial R-modules; i.e., the case $\mathcal{C} = \mathrm{Mod}(R)$. Attached to a simplicial R-module, we have its associated *normalized chain complex* $N(X)$, defines by

$$N(X)_n = \cap_{i=1}^n \ker(d_i : X_n \to X_{n-1}) \subseteq X_n$$

with differential $\partial : N(X)_n \to N(X)_{n-1}$ given as the restriction of d_0. The simplicial identities imply that $N(X)$ really is a chain complex.

There is also a functor taking non-negative chain complexes of R-modules to simplicial modules: Given such a complex

$$\mathbb{M} = (\cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0 \to 0)$$

we set

$$K(\mathbb{M})_{[n]} = \bigoplus_{f:[n] \twoheadrightarrow [d]} M_d$$

I omit the rules for the face and degeneracy maps, but give an example:

Example 4.2. If $\mathbb{F} = (0 \to G \xrightarrow{\alpha} F \to 0)$ in degrees 0 and 1, then $K(\mathbb{F})$ is a "bar construction": $K(\mathbb{F})_n = F \oplus G^{\oplus n}$ with face maps

$$d_j(f, g_1, \ldots, g_n) = \begin{cases} (f + \alpha(g_1), g_2, \ldots, g_n) & \text{for } j = 0, \\ (f, g_1, \ldots, g_j + g_{j+1}, \ldots, g_n) & \text{for } 1 \leq j \leq n-1, \text{ and} \\ (f, g_1, \ldots, g_{n-1}) & \text{for } j = n. \end{cases}$$

The key point about the constructions presented here is:

Theorem 4.3 (Dold-Kan)**.** [Dol58] *The functors N and K are mutually inverse, exact equivalences joining the category of non-negative chain complexes of R-modules with the category of simplicial R-modules. Moreover, chain homotopies correspond to simplicial homotopies.*

4.1.1. *Extending functors to complexes.* Given a (not necessarily additive!) functor T sending finite free modules to finite free modules, we extend it to a functor on non-negative free complexes by translating to the simplicial setting. In detail, given a finite free complex $\mathbb{F}$ with $F_i = 0$ for $i < 0$, we set

$$\tilde{T}(\mathbb{F}) := N(T_*(K(\mathbb{F})))$$

where T_* denotes the application of T degreewise to a simplicial module: $T_*(X_\bullet) : [n] \mapsto T(X_n)$.

Corollary 4.4. *The functor $\tilde{T}$ preserves chain homotopies.*

Proof. The crucial point here is that a simplicial homotopy (see (4.1)) involves only equalities of compositions, and *not* addition or subtraction (in contrast with a chain homotopy). Thus, every functor preserves simplicial homotopies, and the corollary follows from the Dold-Kan theorem. □

Here are some additional properties:

Proposition 4.5. [DP61, 4.7 and Hilfssatz 4.23] *Assume T is a polynomial functor of degree n defined on finite rank free R-modules, and let $\tilde{T}$ denote its extension to non-negative, finite free complexes using the Dold-Kan Theorem. Then*

(1) *If $\mathbb{F}$ is a finite free complex concentrated in degree $[0, m]$, then $\tilde{T}(\mathbb{F})$ is a finite free complex concentrated in degrees $[0, n \cdot m]$.*
(2) *The assignment $T \mapsto \tilde{T}$ is exact: if $0 \to T' \to T \to T'' \to 0$ is an exact sequence of natural transformations of such functors, then so is $0 \to \tilde{T}' \to \tilde{T} \to \tilde{T}'' \to 0$.*
(3) *If $n = 1$ (i.e., T is additive), then $\tilde{T}(\mathbb{F}) \cong T^{naive}(\mathbb{F}) := (0 \to T(F_m) \xrightarrow{T(\partial_m)} \cdots \xrightarrow{T(\partial_1)} T(F_0) \to 0)$.*
(4) *If T "localizes well" (e.g., T^n, Λ^n, S^n, and* Frob*), then $\tilde{T}$ preserves support, in the sense that $\mathbb{F}_\mathfrak{p} \sim 0 \Rightarrow \tilde{T}(\mathbb{F})_\mathfrak{p} \sim 0$. In particular, if the homology of $\mathbb{F}$ has finite length, then so does the $\tilde{T}(\mathbb{F})$.*

We will apply this construction using the examples $T = T^n$ (i.e., $T(F) = \overbrace{F \otimes_R \cdots \otimes_R F}^{n \text{ times}}$), $T = \Lambda^n_R$, $T = \mathrm{S}^n_R$, and, when $\mathrm{char}(R) = p > 0$, $T = \mathrm{Frob}$, extension of scalars along the Frobenius map $\phi : R \xrightarrow{r \mapsto r^p} R$. Recall that each of these is a polynomial functor (and the last one is additive).

Example 4.6. It so happens that T^n and $\widetilde{T^n}$ concide up to natural quasi-isomorphism:

$$T^n(\mathbb{F}) \sim \widetilde{T^n}(\mathbb{F})$$

for any finite free complex $\mathbb{F}$. (But they are *not* isomoprhic.) This equivalence is given by the shuffle map and the Alexander-Whitney map; see [BMTW17, §5] for more details.

Example 4.7. In general, $\Lambda^n(\mathbb{F})$ and $\widetilde{\Lambda^n}(\mathbb{F})$ fail to be quasi-isomorphic, and likewise for $\mathrm{S}^n(\mathbb{F})$ and $\widetilde{\mathrm{S}^n}(\mathbb{F})$ fail to be quasi-isomorphic. See Exercise 2.20.

Example 4.8. When $\mathrm{char}(R) = p > 0$, Frob is additive and thus $\widetilde{\mathrm{Frob}} \cong \mathrm{Frob}$. We note that if

$$\mathbb{F} = (\cdots \xrightarrow{B} R^n \xrightarrow{A} R^m \to 0)$$

for matrices $A, B, \cdots$, then

$$\mathrm{Frob}(\mathbb{F}) \cong (\cdots \xrightarrow{B^{[p]}} R^n \xrightarrow{A^{[p]}} R^m \to 0)$$

where $A^{[p]}, B^{[p]}, \ldots$ denotes raises each entry of these matrices to the p-th power.

The following proposition summarizes what we need from the Dold-Kan Theorem:

Proposition 4.9. *For any R and finite free complex of R-modules $\mathbb{F}$, we have a natural short exact sequence*

$$0 \to \widetilde{\Lambda^2}(\mathbb{F}) \to \widetilde{T^2}(\mathbb{F}) \to \widetilde{S^2}(\mathbb{F}) \to 0 \tag{4.10}$$

of complexes, a natural quasi-isomorphism

$$\widetilde{T^2}(\mathbb{F}) \sim \mathbb{F} \otimes_R \mathbb{F}, \tag{4.11}$$

and hence a natural long exact sequence

$$\cdots \to H_j\widetilde{\Lambda^2}(\mathbb{F}) \to H_j(\mathbb{F} \otimes_R \mathbb{F}) \to H_j\widetilde{\mathrm{S}^2(\mathbb{F})} \xrightarrow{\partial} H_{j-1}\widetilde{\Lambda^2}(\mathbb{F}) \to \cdots \tag{4.12}$$

of R-modules.

When $\operatorname{char}(R) = 2$*, we also have a natural short exact sequence*

$$0 \to \widetilde{\mathrm{Frob}}(\mathbb{F}) \to \widetilde{S^2}(\mathbb{F}) \to \widetilde{\Lambda^2}(\mathbb{F}) \to 0 \tag{4.13}$$

of complexes, a natural isomorphism

$$\widetilde{\mathrm{Frob}}(\mathbb{F}) \cong \mathrm{Frob}(\mathbb{F}), \tag{4.14}$$

and hence a natural long exact sequence

$$\cdots \to H_j \operatorname{Frob}(\mathbb{F}) \to H_j\widetilde{\mathrm{S}^2}(\mathbb{F}) \to H_j\widetilde{\Lambda^2}(\mathbb{F}) \xrightarrow{\partial} H_{j-1} \operatorname{Frob}(\mathbb{F}) \to \cdots . \tag{4.15}$$

of R-modules.

Proof. The exact sequences of complexes (4.10) and (4.13) follow from Proposition 4.5 (2), using the exact sequences given in (2.11) and Proposition 2.14. The quasi-isomorphism (4.11) was discussed in Example 4.6 and the isomorphism (4.14) holds since Frob is additive (see Example 4.8). The existence of the two long exact sequences follows. □

Remark 4.16. Though not really needed in these notes, the Dold-Kan Theorem allows us to deinfe the "non-naive" Adams operations ψ^k on $K_0^{fl}(R)$, and more generally on $K_0^Z(R)$, without any restriction on the characteristic. For instance, given a finite free complex $\mathbb{F}$ concentrated in non-negative degrees, we set

$$\psi^2(\mathbb{F}) := [\widetilde{\mathrm{S}^2}(\mathbb{F})] - [\widetilde{\Lambda^2}(\mathbb{F})].$$

For complexes with negative components, one applies this formula after shifting it and introducing an appropriate sign.

The operators ψ^k have all the same formal properties as their naive counterparts do, but are defined in full generality. For instance, if R is regular local ring or, more generally, a complete intersection, then $\psi^k \circ \chi = 2^d \cdot \chi$ on $K_0^{\mathrm{fl}}(R)$.

See [GS87] for more details.

4.2. **Toral Rank Conjectures in char 2.** In this section I discuss joint work with Keller VandeBogert on rank conjectures for local ring R with $\mathrm{char}(R) = 2$.

Theorem 4.17. [VW24] *Suppose R is a local ring of characteristic 2 and dimension d. If $\mathbb{F}$ is a finite free complex of the form*

$$0 \to F_{d+1} \to \cdots \to F_0 \to 0$$

having non-zero, finite length homology, then $\mathrm{rank}(\mathbb{F}) \geq 2^d$*. In particular, the Total Rank Conjecture for holds for R.*

Remark 4.18. The Theorem also shows that the kinds of counter-examples for the Total Rank Conjecture for Complexes found by Iyengar and myself do not exist in char 2.

Proof. It is not difficult to reduce to the case when R is complete with perfect residue field (but I omit the details) and so let us assume R has these properties.

I will prove the result in detail in the very special (but still very interesting!) case in which R is regular. Since we assume R is complete with perfect residue field, we have $R = k[[x_1, \dots, x_d]]$ with k perfect. After giving the proof in this case, I will then indicate roughly how the proof generalizes.

The fact that R is regular, complete, with perfect residue field gives several desirable properties:

- The homology of $\mathbb{F}$ must lie in degrees 0 and 1 only. (This is true more generally for any CM ring.)
- The Frobenius map is flat. (This holds only when R is regular by a theorem of Kuntz [Kun69]).
- We have $\mathrm{length}(\mathrm{Frob}(M)) = 2^d \,\mathrm{length}(M)$ for any R-module M of finite length. (This uses that k is perfect, so that the Frobenius endomorphisms is an automorphism on the coefficients and sends x_i to x_i^p. Thus $\mathrm{Frob}(k) \cong R/(x_1^2, \dots, x_d^2)$, which is easily seen to have dimension 2^d, and the general result follows.)

These three properties give that

(4.19)

$$H_i(\mathrm{Frob}(\mathbb{F})) = 0 \text{ for } i \geq 2 \text{ and } \mathrm{length}\, H_i(\mathrm{Frob}(\mathbb{F})) = 2^d \cdot \mathrm{length}\, H_i(\mathbb{F}) \text{ for } i = 0, 1.$$

Since $\mathrm{rank}(\mathbb{F})h(\mathbb{F}) \geq h(\mathbb{F} \otimes_R \mathbb{F})$ by Lemma 3.5, we just need to prove

$$h(\mathbb{F} \otimes_R \mathbb{F}) \geq 2^d \cdot h(\mathbb{F}). \tag{4.20}$$

We establish (4.20) by playing the long exact sequences in homology (4.12) and (4.15) off of each other. We note in passing that (4.12) exists also when $\mathrm{char}(k) \neq 2$ (and in that case we may use the "naive" versions Λ^2 and S^2). The boundary map ∂ is zero in that case but, importantly, when $\mathrm{char}(k) = 2$, this boundary map is often non-zero. (If it were zero the proof in the case $\mathrm{char}(k) \neq 2$ would apply.)

By (4.19), $H_i(\mathrm{Frob}(\mathbb{F})) = 0$ for $i \geq 2$, and hence (4.13) gives isomorphisms

$$H_j(\widetilde{\mathrm{S}^2}(\mathbb{F})) \cong H_j(\widetilde{\Lambda^2}(\mathbb{F})) \text{ for all } j \geq 3 \tag{4.21}$$

and an exact sequence

$$\begin{aligned} 0 \to H_2(\widetilde{\mathrm{S}^2}(\mathbb{F})) \to H_2(\widetilde{\Lambda^2}(\mathbb{F}) \to \mathrm{Frob}(H_1(\mathbb{F})) \to H_1(\widetilde{\mathrm{S}^2}(\mathbb{F})) \to H_1(\widetilde{\Lambda^2}(\mathbb{F}) \\ \to \mathrm{Frob}(H_0(\mathbb{F})) \to H_0(\widetilde{\mathrm{S}^2}(\mathbb{F})) \to H_0(\widetilde{\Lambda^2}(\mathbb{F}) \to 0. \end{aligned} \tag{4.22}$$

From (4.12) we obtain the first equality in

$$
\begin{aligned}
h(\mathbb{F}\otimes_R \mathbb{F}) &= \sum_j \operatorname{length}(\ker(H_j\widetilde{\mathrm{S}^2}(\mathbb{F}) \xrightarrow{\partial} H_{j-1}\widetilde{\Lambda^2}(\mathbb{F}))) + \sum_j \operatorname{length}(\operatorname{coker}(H_j\widetilde{\mathrm{S}^2}(\mathbb{F}) \xrightarrow{\partial} H_{j-1}\widetilde{\Lambda^2}(\mathbb{F}))) \\
&\geq \sum_j \left| h_j\widetilde{\mathrm{S}^2}(\mathbb{F}) - h_{j-1}\widetilde{\Lambda^2}(\mathbb{F})) \right| \\
&\geq h_0\widetilde{\mathrm{S}^2}(\mathbb{F}) \\
&\quad + h_1\widetilde{\mathrm{S}^2}(\mathbb{F}) - h_0\widetilde{\Lambda^2}(\mathbb{F})) \\
&\quad - h_2\widetilde{\mathrm{S}^2}(\mathbb{F}) + h_1\widetilde{\Lambda^2}(\mathbb{F})) \\
&\quad - h_3\widetilde{\mathrm{S}^2}(\mathbb{F}) + h_2\widetilde{\Lambda^2}(\mathbb{F})) \\
&\quad - h_4\widetilde{\mathrm{S}^2}(\mathbb{F}) + h_3\widetilde{\Lambda^2}(\mathbb{F})) \\
&\quad - \cdots \\
&= h_0\widetilde{\mathrm{S}^2}(\mathbb{F}) + h_1\widetilde{\mathrm{S}^2}(\mathbb{F}) - h_0\widetilde{\Lambda^2}(\mathbb{F}) - h_2\widetilde{\mathrm{S}^2}(\mathbb{F}) + h_1\widetilde{\Lambda^2}(\mathbb{F}) + h_2\widetilde{\Lambda^2}(\mathbb{F}).
\end{aligned}
\tag{4.23}
$$

The inequalities are elementary, and last equality is a consequence of (4.21). By (4.22) we have

$$h_0 \operatorname{Frob}(\mathbb{F}) \leq h_0\widetilde{\mathrm{S}^2}(\mathbb{F}) - h_0\widetilde{(}\Lambda^2(\mathbb{F})) + h_1\widetilde{(}\Lambda^2(\mathbb{F}))$$

and

$$h_1 \operatorname{Frob}(\mathbb{F}) \leq h_2\widetilde{(}\Lambda^2(\mathbb{F})) - h_2\widetilde{\mathrm{S}^2}(\mathbb{F}) + h_1\widetilde{\mathrm{S}^2}(\mathbb{F}).$$

Combining these with (4.23) gives

$$h(\operatorname{Frob}(\mathbb{F})) \leq h(\mathbb{F}\otimes_F \mathbb{F})$$

Finally, by (4.19) we have $h(\operatorname{Frob}(\mathbb{F})) = 2^d h(\operatorname{Frob}(\mathbb{F})$, which gives (4.20) and completes the proof in the regular case.

I give only the vague idea of the proof for the general case: for an arbitrary R (still assumed complete with perfect residue field), the key properties (4.19) that hold in the regular case remain valid asymptotically, upon repeated application of the Frobenius endomorphism. By taking suitable limits, essentially the same argument given here in the regular case works in general. □

4.3. **Carlsson's Conjecture in char 2.** The same proof technique used in the previous section gives a mildly interesting result regarding Carlsson's conjecture:

Theorem 4.24. [VW24] *Let*

$$\mathbb{F} = (0 \to F_m \to \cdots \to F_1 \to F_0 \to 0)$$

be a non-trivial, finite free complex over $R = k[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ *with* $\operatorname{char}(k) = 2$*. If* $m \leq 1$ *then* $h(\mathbb{F}) := \dim_k H(\mathbb{F}) \geq 2^n$ *and if* $m = 2$ *then* $h(\mathbb{F}) \geq 2^{n-1}$*.*

Remark 4.25. When $m = 0$, this is obvious, since $\dim_k(R) = 2^n$. When $m = 1$ this was proven originally by Adem-Swan [AS95, Corollary 2.1].

Sketch of Proof. I'll sketch the proof when $m = 2$. The key inequality is

$$\operatorname{rank}(\mathbb{F}) \cdot h(\mathbb{F}) \geq h_1(\operatorname{Frob}(\mathbb{F})). \tag{4.26}$$

To deduce the result from (4.26), we observe that for any finite free complex $\mathbb{F}$, if $\chi(\mathbb{F}) \neq 0$, then $h(\mathbb{F}) \geq 2^d$, since $h(\mathbb{F}) \geq |\chi(\mathbb{F})|$ and $\chi(\mathbb{F}) = \sum_i (-1)^i 2^d \operatorname{rank}(F_i)$ is a multiple of 2^d. So, we may assume $\chi(\mathbb{F}) = 0$; i.e. that $h_1(\mathbb{F}) = h_0(\mathbb{F}) + h_2(\mathbb{F})$ and $\operatorname{rank}(F_1) = \operatorname{rank}(F_0) + \operatorname{rank}(F_2)$. We may also assume that $\mathbb{F}$ is minimal. Since the Frobenius map sends x_i to 0 for all i and $\mathbb{F}$ is minimal, $\operatorname{Frob}(\mathbb{F})$ has trivial differential, and hence $h_1(\operatorname{Frob}(\mathbb{F})) = \dim_k F_1 = 2^n \cdot \operatorname{rank}(F_1)$. From (4.26) we thus get

$$h(\mathbb{F}) \geq 2^n \frac{\operatorname{rank}(F_1)}{\operatorname{rank}(\mathbb{F})} = 2^n \frac{\operatorname{rank}(F_1)}{\operatorname{rank}(F_0) + \operatorname{rank}(F_1) + \operatorname{rank}(F_2)} = 2^n \frac{\operatorname{rank}(F_1)}{2\operatorname{rank}(F_1)} = 2^{n-1}.$$

To establish (4.26), since $m = 2$, we have $H_i(\operatorname{Frob}(\mathbb{F})) = 0$ for $i \geq 3$ and thus the long exact sequence (4.15) gives $H_i(\widetilde{\mathrm{S}^2}(\mathbb{F})) \cong H_i(\widetilde{\Lambda^2}(\mathbb{F}))$ for $i \geq 3$. Using this and the long exact sequence (4.12) one may deduce that

$$h(\mathbb{F} \otimes_R \mathbb{F}) \geq h_1(\widetilde{\mathrm{S}^2}(\mathbb{F})) + h_2(\widetilde{\Lambda^2}(\mathbb{F})).$$

Using (4.15) again gives

$$h_1(\widetilde{\mathrm{S}^2}(\mathbb{F})) + h_2(\widetilde{\Lambda^2}(\mathbb{F})) \geq h_1(\operatorname{Frob}(\mathbb{F})).$$

Since $\operatorname{rank}(\mathbb{F}) \cdot h(\mathbb{F}) \geq h(\mathbb{F} \otimes_R \mathbb{F})$ by Lemma 3.5, this establishes (4.26). □

Remark 4.27. The same ideas in this proof show that if $\mathbb{F} = (0 \to F_{d+2} \to \cdots \to F_1 \to F_0 \to 0)$ is a finite free complex over a local ring of dimension d such that $\chi(\mathbb{F}) = 0$, then

$$\operatorname{rank}(\mathbb{F}) \geq 2^{d-1}.$$

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