RANK CONJECTURES ACROSS ALGEBRA AND TOPOLOGY
MASTERCLASS AT THE UNIVERSITY OF COPENHAGEN
24-28 JUNE 2024

This document contains exercises for all of the lecture series. They are roughly in order of complexity of the machinery involved: the first few pages can tackled in the first exercise session, and later exercises might require material from later lectures. Other than this, the order is essentially random and you should feel free to work on the problems that look most interesting to you.

Exercise 1 (Zoller)**.** Find a free action of a torus T on a compact space X such that the inclusion of any orbit induces the trivial map $H(X;\mathbb{Q}) \to H(T;\mathbb{Q})$.

Exercise 2 (Yalçın)**.** Using the definition of group cohomology, prove that for every kG-module M, $H^0(G;M) \cong M^G := \{m \in M \mid gm = m \text{ for all } g \in G\}$.

Exercise 3 (Walker)**.** For a local ring R with maximal ideal $\mathfrak{m}$, prove that $x_1, \ldots, x_c \in \mathfrak{m}$ is a regular sequence if and only if $\mathrm{Kos}_R(x_1, \ldots, x_c)$ is exact everywhere except in degree 0, and hence is a bounded free resolution of $R/(x_1, \ldots, x_c)$.

Hint: Use that $\mathrm{Kos}_R(x_1, \ldots, x_{i+1})$ is the mapping cone of multiplication by x_{i+1} on $\mathrm{Kos}_R(x_1, \ldots, x_i)$.

Exercise 4 (Yalçın)**.** Let $G = \langle g \mid g^n = 1\rangle \cong C_n$ be a cyclic group of order n. Verify that the chain complex

$$(F_*, \varepsilon) : \ldots \longrightarrow kG \xrightarrow{N_G} kG \xrightarrow{1-g} kG \xrightarrow{N_G} kG \xrightarrow{1-g} kG \xrightarrow{\varepsilon} k \longrightarrow 0\,,$$

where $N_G = 1 + g + \cdots + g^{p-1}$ is the norm element and ε is the augmentation map defined by $\varepsilon(\sum_g a_g g) = \sum_g a_g$, is a free resolution for k as a kG-module.

Exercise 5 (Zoller)**.**

(a) Let $T = T^r$ and let $U \subset T$ be a subcircle. Then $H_T(T/U) \cong \mathbb{Q}[x]$ and the map $T/U \mapsto T/T = \{*\}$ induces a surjection $H(BT) = H_T(\{*\}) \to H_T(T/U)$

(b) Let X be a reasonable space (e.g. a finite CW complex) with a T-action. Prove that $\dim_{\mathbb{Q}} H_T(X) = \infty$ if the action is not almost free.

Hint factor the map from part (a) through $H_T(X)$.

Exercise 6 (Zoller)**.** Let X be a finite CW complex with an almost free torus action. Prove that $\chi(X) = 0$.

Hint: One way to approach this is to use the Serre spectral sequence for the fibration $X \to X_T \to BT$.

Note: More generally it is true that $\chi(X) = \chi(X^T)$ but we will not discuss all of the the tools needed for this.

Exercise 7 (Walker)**.** Prove $[\mathbb{F}] + [\mathbb{F}'] = [\mathbb{F} \oplus \mathbb{F}']$ and $[\mathbb{F}] = -[\Sigma\mathbb{F}]$ where $\Sigma\mathbb{F}$ is the suspension of $\mathbb{F}$. In particular, conclude that every class in $K_0^{\mathrm{fl}}(R)$ may be represented by a single complex concentrated in non-negative degrees.

Exercise 8 (Walker)**.** If R is a regular local ring, then $\chi\colon K_0^{\mathrm{fl}}(R) \xrightarrow{\cong} \mathbb{Z}$ is an isomorphism. Moreover, the element $1 \in \mathbb{Z}$ corresponds to $[\mathrm{Kos}_R(x_1, \ldots, x_d)]$ for a minimal set of generators $x_1, \ldots, x_d$ of $\mathfrak{m}$.

Exercise 9 (Yalçın)**.** Suppose that $G = \langle g \mid g^p = 1\rangle \cong C_p$ is a finite cyclic group of order p, and k is a field of char p. Let $F_* \to k$ be the periodic free kG-resolution of k from the previous exercise. Show that the map $\delta\colon F_* \to F_* \otimes_k F_*$ defined by

$$\delta_{m,n}(1) = \begin{cases} 1 \otimes 1 & \text{if } m \text{ is even} \\ 1 \otimes g & \text{if } m \text{ is odd}, n \text{ is even} \\ \sum_{0 \le i < j \le n-1} g^i \otimes g^j & \text{if } m \text{ is odd}, n \text{ is odd} \end{cases}$$

is a diagonal approximation. Using this diagonal approximation, prove that there are isomorphism of k-algebras

$$H^*(C_p;k) \cong \begin{cases} k[x] & \text{if } p = 2 \\ \wedge_k(y) \otimes k[x] & \text{if } p > 2 \end{cases}$$

where $|x| = 1$ if $p = 2$, and $|y| = 1$, $|x| = 2$ if $p > 2$.

Note: the next two exercises are similar and you should pick which one looks more fun to you and just do that. As a hint, the first one is best done using the Serre spectral sequence, and the second one can be done using the Mayer–Vietoris exact sequence.

Exercise 10 (Zoller)**.** Consider the embedding

$$f\colon X = \mathbb{C}P^2 \vee \mathbb{C}P^2 \to \mathbb{C}P^\infty \vee \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty = BT^2.$$

Now pull back the unversal T^2-bundle $T^2 \to ET \to BT$ along f to obtain a principal T-bundle

$$T \to Y \to X.$$

(a) Compute that the betti numbers $b_0, \ldots, b_4$ of Y are given by $1, 0, 0, 3, 2$ and vanish in higher degrees.
(b) Conclude that the cohomology ring of a space with a free T^r action does not necessarily contain an exterior algebra on r generators.

Exercise 11 (Zoller)**.** Consider the T^2 action on $X_1 = S^1 \times S^3 \subseteq \mathbb{C}^3$ given by $(s,t)\cdot(u,v,w) = (su, tv, tw)$, as well as the T^2 action on $X_2 = S^3 \times S^1 \subseteq \mathbb{C}^3$ given by $(s,t)\cdot(u,v,w) = (su, sv, tw)$.

Let $x_0 = (1,0,1) \in X_1 \cap X_2$ and let Y be the T^2-space which arises by gluing X_1 and X_2 along the orbit $T^2 \cdot x_0$.
(a) Prove that the T^2-action on Y is free.
(b) Compute that the betti numbers $b_0, \ldots, b_4$ of Y are given by $1, 0, 0, 1, 2$ and vanish in higher degrees.
(c) Conclude that the cohomology ring of a space with a free T^r action does not necessarily contain an exterior algebra on r generators.

Exercise 12 (Yalçın)**.** Give a full proof for the following proposition: If k is a field of characteristic p, then

$$H^*((C_p)^r; k) \cong \begin{cases} k[x_1, \ldots, x_r] & \text{if } p = 2 \\ \wedge_k(y_1, \ldots, y_r) \otimes k[x_1, \ldots, x_r] & \text{if } p > 2 \end{cases}$$

where $\deg x_i = 1$ if $p = 2$, and $\deg y_i = 1$, $\deg x_i = 2$ if $p > 2$.

Exercise 13 (Walker)**.** If R is a regular local ring, then $G_0(R) \cong \mathbb{Z}$, generated by the class of $[R]$.

Exercise 14 (Walker)**.** Show $\chi(-,-)$ preserves the relations in each argument and thus extends to a bi-linear pairing

$$\chi(-,-)\colon K_0^{\mathrm{fl}}(R) \otimes_{\mathbb{Z}} G_0(R) \to \mathbb{Z},$$

which we also write as $\chi(-,-)$.

Exercise 15 (Zoller)**.** Let (A,d) be a $\mathbb{Q}$-cdga which is 1-connected, i.e. $H^0(A,d) = \mathbb{Q}$ and $H^1(A,d) = 0$. Prove that (A,d) has a minimal Sullivan model by using the following inductive approach:

Fix n and assume we have constructed $(\Lambda V, d)$ with V generated in degrees $< n$ as well as $\varphi\colon (\Lambda V, d) \to (A,d)$ such that $H^k(\Lambda V, d) \to H^k(A,d)$ is an isomorphism for $k < n$ and is injective for $k = n$. Now generators in degree n are introduced in the following way

- introduce a vector space U in degree n, set $d|U = 0$ and extend φ to U such that $\varphi^*\colon H^n(\Lambda(V \oplus U), d) \to H^n(A,d)$ becomes an isomorphism.
- introduce new generators W in degree n and define $d|_B$ such that it kills $\ker(H^{n+1}(\Lambda(V \oplus U), d) \to H^{n+1}(A,d)$. Now extend φ to W.

Note: this exercise needs some familiarity with Sullivan models, especially how to inductively define a map out of a Sullivan model by specifying it on the generators. If you want help on how this works, chat to a TA!

Exercise 16 (Zoller)**.** Using the above algorithm, compute the minimal model of S^n (this takes a different form depending on whether n is even or odd).

Hint: You will not need any information about $A_{PL}(S^n)$ except for its cohomology. So in fact this shows that all cdgas with this cohomology have the same minimal model (this very useful property is called "intrinsic formality").

Exercise 17 (Yalçın)**.** Show that if X is an admissible G-simplicial complex then for each $H \leq G$, $|X^H| = |X|^H$. Prove that if X is a G-simplicial complex, then the barycentric subdivision $sd(X)$ is admissible. Also show that if X is admissible, then $sd(X)$ is regular.

Exercise 18 (Walker)**.** Verify the following sequence is exact

$$0 \to \Lambda^2(F) \xrightarrow{v\wedge w \mapsto v\otimes w - w \otimes v} T^2(F) \xrightarrow{\text{can}} \mathrm{S}^2(F) \to 0\,.$$

Exercise 19 (Yalçın)**.** Let X be the 1-dimensional simplicial complex whose vertex set is $V = \{a,b,c,d\}$ and whose 1-simplices are given by $\{a,b\}$, $\{b,c\}$, $\{c,d\}$, and $\{d,a\}$. Choose an orientation for each simplex and write the simplicial chain complex of X. Show that $H_1(X;k) \cong H_0(X;k) \cong k$.

Exercise 20 (Yalçın)**.** Show that if $0 \to A \to F_{k-1} \to \cdots \to F_0 \to B \to 0$ is an exact sequence of kG-modules where $F_0, \ldots, F_{k-1}$ are free kG-modules, then $H^i(G;B) \cong H^{i+k}(G;A)$ for all $i \geq 1$.

Exercise 21 (Walker)**.** Let $\mathbb{F} = \mathrm{Kos}_R(x)$; that is, $\mathbb{F} = (0 \to R\cdot b \xrightarrow{\partial} R \cdot a \to 0)$ for formal symbols a and b of degrees 0 and 1, with $\partial(b) = xa$. Then

$$T^2(\mathbb{F}) = (0 \to F_2 \to F_1 \to F_0 \to 0)$$

with $F_2 = R\cdot(b \otimes b)$, $F_1 = R\cdot(a\otimes b) \oplus R\cdot(b \otimes a)$ and $F_0 = R\cdot(a \otimes a)$, with the action of τ given by $b\otimes b \mapsto -b\otimes b$, $a\otimes b \mapsto b \otimes a$, and $a \otimes a \mapsto a \otimes a$.

(a) Show that if 2 is invertible in R, then a basis of the free module underlying $\mathrm{S}^2_{\text{naive}}(\mathbb{F})$ is $\{\frac{1}{2}(a\otimes b+b\otimes a), a\otimes a\}$ and hence
$$\mathrm{S}^2_{\text{naive}}(\mathbb{F}) \cong (0 \to R \xrightarrow{x} R \to 0) = \mathrm{Kos}_R(x).$$
(b) Show that when 2 is a invertible, $\Lambda^2_{\text{naive}}(\mathbb{F})$ has basis $\{b\otimes b, a\otimes b - b\otimes a\}$, and thus
$$\Lambda^2_{\text{naive}}(\mathbb{F}) \cong (0 \to R \xrightarrow{x} R \to 0) \text{ with } R \text{ in degrees 1 and 2;}$$
i.e., $\Lambda^2(\mathbb{F}) \cong \Sigma^1\mathrm{Kos}_R(x)$.
(c) Show that if $\mathrm{char}(R) = 2$, then
$$\mathrm{S}^2_{\text{naive}}(\mathbb{F}) \cong (0 \to R \xrightarrow{x} R \xrightarrow{2} R \to 0) = (0 \to R \xrightarrow{x} R \xrightarrow{0} R \to 0).$$
By taking $x = 1$, conclude that $\mathrm{S}^2_{\text{naive}}$ does not preserve homotopies in this case.

Exercise 22 (Zoller)**.**
(a) Let a, b be of degree 3, c of degree 5 and consider $A = \Lambda(a, b, c)$ with $da = 0 = db$ and $dc = ab$. Compute $H(A, d)$.
(b) Compute the minimal model of the cdga $(H(A, d), 0)$ with trivial differential up to degree 11. (The algorithm does not terminate after finitely many steps. In particular the minimal model is not (A, d) but another minimal cdga with the same cohomology).

Exercise 23 (Yalçın)**.** Consider the action of the symmetric group $G = S_3$ on the simplicial complex X with vertex set $V = \{1, 2, 3\}$ and edge set
$$S = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$$
given by the permutation of the vertices with the natural action. The realization of X is the boundary of a triangle with corners labeled as 1, 2, 3. Observe that this action is not admissible but the G-action on the barycentric subdivision $Y = sd(X)$ is admissible. Write the chain modules and boundary maps for the simplicial chain complex of X and Y.

Exercise 24 (Walker)**.** Show that if $0 \to \mathbb{F}' \to \mathbb{F} \to \mathbb{F}'' \to 0$ is a short exact sequence (not necessarily split) of finite free complexes, then
$$\psi^2(\mathbb{F}) = \psi^2(\mathbb{F}') + \psi^2(\mathbb{F}'').$$

Exercise 25 (Yalçın)**.** Prove that if G is a finite group and k has characteristic p, then $kG^* \cong kG$. Using this, conclude that if M is a finitely-generated kG-module, then M is projective if and only if M is injective.

Exercise 26 (Walker)**.** Go to Mark Walker's notes and admire Theorem 3.8 (about the Toral Rank Conjecture for formal dg modules). Note that alas, the proof there has only been sketched. Complete the details of this proof by making suitable modifications to the proof of Theorem 3.1.

Exercise 27 (Yalçın)**.** Show that if p is odd, and X is a finite-dimensional free C_p-CW-complex X such that $H_*(X; \mathbb{F}_p) \cong H_*(S^n; \mathbb{F}_p)$, then n is an odd number.

Exercise 28 (Walker)**.** Prove the following lemma from the notes using one of the two approaches below: For any commutative ring R, any finite free complex $\mathbb{F}$, and any bounded complex $\mathbb{M}$ of R-modules having finite length homology, we have
$$h(\mathbb{F} \otimes_E \mathbb{M}) \le \mathrm{rank}(\mathbb{F})h(\mathbb{M}).$$
(a) Exploit the spectral sequence
$$H_i(\mathbb{F} \otimes_R H_j\mathbb{M}) \Rightarrow H_{i+j}(\mathbb{F} \otimes_R \mathbb{M})$$
using also that $h(\mathbb{F} \otimes N) \le \mathrm{rank}(\mathbb{F})\mathrm{length}(N)$ for any module N having finite length.
(b) Alternatively, proceed by induction on $h(\mathbb{M})$. For the case $h(\mathbb{M})$ use that we may assume $\mathbb{M} = k$.

Exercise 29 (Zoller)**.**
(a) Let (A, d) be a cdga whose cohomology is a free cdga ΛV. Show that $(\Lambda V, 0)$ is the minimal model of (A, d)
(b) Let $H \subset G$ be simply connected compact Lie groups. Prove that the homogeneous space G/H is rationally elliptic.

Hint: You may use that the rational cohomology of a compact Lie group is an exterior algebra.

Exercise 30 (Walker)**.** Prove that when R is a domain, the map taking a finitely generated R-module to its rank induces an isomorphism $G_0(R)_{(d)} \xrightarrow{\cong} \mathbb{Q}$ using the second property in Theorem 3.10 in Mark's notes, and the following "localization" exact sequence for G-theory: the sequence
$$\bigoplus_{f\neq 0} G_0(R/f) \to G_0(R) \to G_0(F) \to 0$$
(with the maps being the canonical ones) is exact, where F is the field of factions of R.

Exercise 31 (Yalçın)**.**

(a) Show that if G is a finite group which acts freely and cellularly on a finite-dimensional CW-complex X such that
$$H_*(X;\mathbb{F}_p) \cong H_*(S^n \times S^n; \mathbb{F}_p),$$
then there is a long exact sequence
$$\cdots \to H^i(G;k) \to H^{i+n+1}(G;M) \to H^{i+2n+2}(G,k) \to H^{i+1}(G;k) \to H^{i+n+2}(G,M) \to \cdots$$
where $M = H^n(X;k)$.

(b) Using the exact sequence in part (a) and using Exercise 12, prove Conner's theorem: if G acts freely on a finite-dimensional CW-complex X which has mod-p homology of $S^n \times S^n$, then G does not include $\mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p$ as a subgroup.

Exercise 32 (Zoller)**.** Let $R = \Lambda(x_1, \dots, x_r)$ and set $K_n = (R \otimes \Lambda(s_1^n, \dots, s_r^n), d)$ with s_i^n of degree $2n+1$ and $ds_i = x_i^{n+1}$. For $r = 4$, construct a nonzero degree 0 morphism of dg R-modules $K_n \to K_0$ which is not injective.

Exercise 33 (Walker)**.** Show that if $\mathbb{F}$ is tiny and M is an MCM module, then $H_i(\mathbb{F} \otimes_R M) = 0$ for all $i \neq 0$.

Hint: Consider the bicomplex $\mathbb{F} \otimes_R \mathcal{C} \otimes_R M$ where $\mathcal{C}$ is the "algebraist's" Čech complex on a system of parameters $x_1, \dots, x_d \in \mathfrak{m}$ of R:
$$\mathcal{C} = (0 \to R \to \bigoplus_i R\left[\frac{1}{x_i}\right] \to \bigoplus_{i<j} R\left[\frac{1}{x_i x_j}\right] \to \cdots \to R\left[\frac{1}{x_1 \cdots x_d}\right] \to 0).$$
You may use that M is MCM if and only if $\mathcal{C} \otimes_R M$ has homology only in the right-most position. (The homology of $\mathcal{C} \otimes_R M$ gives $H^*_{\mathfrak{m}}(M)$, the local cohomology of M supported at the maximal ideal.)

Exercise 34 (Yalçın)**.** Let $G \cong (\mathbb{Z}/2)^r$ be an elementary abelian 2-group of rank r, and let X be a finite free G-CW-complex such that $H^*(X;\mathbb{F}_2) \cong S^n \times S^m$ with $1 \leq n < m$. Analyse the differentials on the Serre spectral sequence for the Borel fibration $X \to EG \times_G X \to BG$ and conclude that $r \leq 2$ (see section 7 of *Steenrod closed parameter ideals in the mod-2 cohomology of A_4 and $SO(3)$* by Rüping, Stephan and Yalçın for a similar analysis).

Exercise 35 (Walker)**.** Disprove the total rank conjecture for complexes using the construction in section 3.4 in Mark's notes:

Let $R = k[\![x_1, \dots, x_d]\!]$ and assume $\operatorname{char}(k) = 0$ (or $\operatorname{char}(k) > \frac{d}{4} + \frac{1}{2}$). Let $\mathbb{K} = \operatorname{Kos}_R(x_1, \dots, x_d)$ be the Koszul complex of R (with exterior generators $e_1, \dots, e_d$ corresponding to $x_1, \dots, x_d$), and then define $\mathbb{F} = \operatorname{cone}(q)$ where q is the degree -2 chain map
$$q = e_1^* e_2^* + e_3^* e_4^* + \cdots + e_{d-1}^* e_d^* \colon \mathbb{K} \to \mathbb{K}.$$
Let $\overline{\mathbb{F}}$ be the minimal complex homotopy equivalent to $\mathbb{F}$. Prove that:

(1) $\overline{\mathbb{F}}$ is a non-trivial finite free complex with $H_i(\overline{\mathbb{F}}) = H_i(\mathbb{F}) = k$ for $i = 0$ or $i = 1$ and $H_i(\overline{\mathbb{F}}) = H_i(\mathbb{F}) = 0$ for all other values of i.
(2) $\mathbb{F}$, and so $\overline{\mathbb{F}}$, is concentrated in degrees $[0, d+1]$.
(3) $\operatorname{rank}(\overline{\mathbb{F}}) = \dim_k H_*(k \otimes_R \mathbb{F}) = \binom{d+2}{\frac{d}{2}+1}$.
(4) Deduce that when $d \geq 8$ and the Total Rank Conjecture for Complexes is false for $R = k[[x_1, \dots, x_d]]$.

You will need to use the following fact taken from *Examples of finite free complexes of small rank and small homology* by Iyengar and Walker: With the notation above, provided $\operatorname{char}(k) = 0$ or $\operatorname{char}(k) > \frac{d}{4} + \frac{1}{2}$, the map
$$q = \sum_{i=1}^{d/2} e_{2i-1}^* e_{2i}^* : \Lambda^j(e_1^*, \dots, e_d^*) \to \Lambda^{j+2}(e_1^*, \dots, e_d^*)$$
has full rank—i.e., it is injective for $j \leq d/2 - 1$ and surjective for $j \geq d/2 - 1$.

Exercise 36 (Walker)**.** Prove the Algebraic Version of Carlsson's Conjecture is false when $n \geq 8$ is even, as follows: Recall $R = k[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$ with $\operatorname{char}(k) = p$. Let $\mathbb{K} = \operatorname{Kos}_R(x_1, \dots, x_n)$.

(a) Show $H_*(\mathbb{K})$ is an exterior algebra over k generated by $H_1(\mathbb{K}) \cong k^n$.

(b) Let $z \in K_2$ be a cycle representing the class of an element $q \in H_2(\mathbb{K})$ as in the previous exercise, and set $\mathbb{F}$ to be the cone of multiplication by z. Show $\mathbb{F}$ is a counter-example by applying the same lemma from Iyengar–Walker as above.

Exercise 37. Explain why the participants of the masterclass on "Exit paths and stratified homotopy types" are worse at football and generally morally inferior to the participants of this masterclass.