The Toral Rank Conjecture

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Abstract

These are notes for a four part mini course on the Toral Rank Conjecture, held in Copenhagen June 24–28 2024. The goal is to briefly review the path from geometry to commutative algebra as well as survey some of the most important results regarding the conjecture. The notes roughly reflect the content of the talks and in particular do not give full proofs of all results but rather aim to illustrate the key ideas and techniques.

1 Introduction

Throughout the notes $H(-)$ denotes singular cohomology. Coefficients are taken in $\mathbb{Q}$ unless stated otherwise. By $T = T^r = S^1 \times \ldots \times S^1$ we will denote the compact torus of dimension r and X will denote a topological space (which will most of the time assumed to be a finite CW-complex).

Definition 1.1. An action of T on X is called (almost) free if for any $x \in X$ the stabilizer $T_x = \{t \in T \mid tx = x\}$ is trivial (finite).

The very naive geometric intuition is that in order for a rotation symmetry of a space to be free, the space needs to rotate around a "hole".

Example 1.2. Consider the action of T^2 on $S^3 \subset \mathbb{C}^2$ given by $(s,t) \cdot (v,w) = (sv, tw)$. We point out that the action is not free since there are two 1-dimensional orbits which are circles around which the action rotates. If we force freeness of the action by removing these orbits around which the action rotates then we generate cohomology (in fact the result is homotopy equivalent to T^2).

The Idea of the toral rank conjecture is that (almost) free torus symmetries force big cohomology. One has $\dim H(T^r) = 2^r$. Also T^r acts freely on itself, setting the bar for a relation between betti numbers and symmetry. As we will see later, some topological requirements on X are necessary for such a connection to hold. We call such a space reasonable and, for the purpose of these notes, take that to mean a finite CW-complexes (although generalizations are possible).

Conjecture 1.3 (Halperin, [12])**.** *Let X be a reasonable space with an almost free T^r-action. Then*

$$\dim H(X) \geq 2^r.$$

This conjecture is known as the Toral Rank Conjecture. We abbreviate it by TrRC (not to be confused with the total rank conjecture about Betti numbers of modules!).

Remark 1.4. If G is a compact Lie group with maximal torus T^r then $\dim H(G) = 2^r$. Hence the TrRC implies the analogous conjecture for all other compact Lie groups: If $G \curvearrowright X$ almost freely then $\dim H(X) \geq \dim H(G)$.

We give a brief (and incomplete!) overview on the conjecture

- It is open! (and over 40 years old)
- Has been solved for some spaces: compact Kähler manifolds [3] (more generally weak Lefschetz type cohomologically symplectic spaces [1]), compact homogeneous spaces [2] and more generally certain 2-stage spaces [15], certain nilmanifolds [7], [6]
- Connections to Walkers solution of the total rank conjecture [18] solve the TrRC under certain formality conditions [5]
- Allday and Puppe prove (see [4]) that in general one has a lower bound of $2r$ and of $2(r+1)$ if $r \geq 3$. Furthermore $H^k(X) \neq 0$ for $r+1$ different values.
- The TrRC holds for $\dim T \leq 3$ or $\dim X \leq 7$.

2 From geometry to algebraic topology

For every topological group G there is a universal principal G-bundle

$$G \to EG \to BG$$

where EG is contractible (unique up to equivariant homotopy equivalence)

Example 2.1. There is a free S^1-action on $S^{2n-1} \subset \mathbb{C}^n$ by diagonal complex multiplication. This induces a free action on $S^\infty = ES^1$ (weak topology). So we have $BS^1 = \mathbb{C}P^\infty$. For the r-torus we consider the r-fold products $ET^r = (S^\infty)^r$, $BT^r = (\mathbb{C}P^\infty)^r$.

Remark 2.2. ET provides a counterexample to the TrRC. It is however not reasonable.

If $G \curvearrowright X$ then consider the diagonal action on $EG \times X$ and consider the *Borel construction* $X_G = (EG \times X)/G$. Then there is a fiber bundle

$$X \to X_G \to BG$$

given by projection onto the first factor. It is called the *Borel fibration*.

Definition 2.3. $H_G(X) := H(X_G)$ is the (Borel) equivariant cohomology.

We collect some properties:

- The map $X_G \to BG$ induces a map $H(BG) \to H_G(X)$
- A G-equivariant map $X \to Y$ induces

$$\begin{array}{ccccc} X & \longrightarrow & X_G & \longrightarrow & BG \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Y_G & \longrightarrow & BG \end{array}$$

  and hence a map $H_G(Y) \to H_G(X)$ of $H(BG)$-algebras.
- $H_G(-)$ inherits all the nice properties from $H(-)$ as a G-invariant open $U \subset X$ induces an open set $U_G \subset X_G$.

Example 2.4.
- $H(BT^r) = H((\mathbb{C}P^\infty)^r) = \mathbb{Q}[x_1, \dots, x_r]$ with x_i in degree 2. We denote this ring by R.

- For single orbits T/U one has

$$(ET \times T/U)/T = ET/U = BU.$$

In particular $H_T(*) = H(T/T) = R$. If U is finite then we have $U =\cong \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_l}$ and $BU \simeq \prod(S^\infty)/\mathbb{Z}_{n_i}$. In this case $H_T(T/U) = \mathbb{Q}$.

- If the action is free (and X is sufficiently nice) then the projection $X_T \to X/T$ is a homotopoy equivalence since it is a fiber bundle with contractible fiber ET. This gives a partial proof of the Theorem below.

Theorem 2.5 (cf. [13])**.** *Let X be a reasonable T-space. Then the action is almost free if and only if* $\dim H_T(X) < \infty$.

Sketch of proof. Since X is reasonable we find a tube around every orbit O, i.e. an open set U which deformation retracts equivariantly onto the orbit. Since the orbit has finite stabilizer we have $H_T(U) = H_T(O) = \mathbb{Q}$ by Example 2.4. As X is compact we can cover X with n such tubes and hence X_T is covered by n open sets with trivial cohomology. Using relative cup products one deduces that the maximal product length in $H_T(X)$ (of positive degree elements) is bounded by n. In particular the image of $R \to H_T(X)$ is finite dimensional. The statement now follows from the fact that $H_T(X)$ is finitely generated over R.

To justify the latter fact one uses the Serre spectral sequence of the Borel fibration. The page$E_2 = R \otimes H(X)$ is finitely generated as an R-mo. Since R is noetherian the subquotients E_r and E_∞ inherit this property. From this it follows that also $H_T(X)$ is finitely generated.

The other direction will be treated in the exercises.

□

This leads us to

Conjecture 2.6 (Version 2)**.** *If $X \to Y \to BT^r$ is a fibration with* $\dim H(Y) < \infty$ *then*

$$\dim H(X) \geq 2^r.$$

As another consequence of Theorem 2.5 we obtain

Corollary 2.7. *If X carries an almost free T^r-action with $r \geq 1$ then $\chi(X) = 0$*

3 From topology to commutative algebra

Our goal is to translate $X \to X_T \to BT$ into algebraic data. To do this we give a (criminally brief) introduction to rational homotopy theory. We refer to [8] for an extensive treatment of the subject. Furthermore [9, Chapter 7] discusses the applications to torus actions in more detail.

Definition 3.1. A differential graded algebra (dga) is a cochain complex (A, d) which is an algebra (over some ground field k) such that $A^p \cdot A^q \subset A^{p+q}$ and d is a derivation, i.e. $d(ab) = da \cdot b + (-1)^|a|a \cdot db$ where $|a|$ denotes the degree ofa homogeneous element a. It is called commutative (cdga) if $xy = (-1)^{|x|\cdot|y|}yx$.

We will furthermore assume all cdgas to be concentrated in nonnegative degrees.

Example 3.2.
- Singular cochains $C^*(X, k)$ with the cup product. Not commutative.
- $H(X, k)$ with trivial differential. Commutative.

- M manifold, $k=\mathbb{R}$: the differential forms $(\Omega(M),d)$ are a cdga.

Definition 3.3. A morphism of (c)dgas $(A,d)\to(B,d)$ is a morphism of algebras which preserves degrees and commutes with d. We call it a quasi isomorphism if it induces an isomorphism $H(A)\to H(B)$. Two (c)dgas are called quasi isomorphic if there is a chain of quasi isomorphisms

$$(A,d)\to\bullet\leftarrow\cdots\leftarrow\bullet\to(B,d).$$

As the cohomology of a space is always commutative it is natural to ask whether one can find a commutative cochain model, i.e. whether $C^*(X,k)$ is quasi isomorphic to a cdga. The answer is in general negative (the Steenrod operations form an obstruction). However one has

Theorem 3.4 ([17])**.** *There is a functor, the polynomial forms,*

$$A_{PL}\colon \mathbf{Top}\to\mathbb{Q}\textbf{-cdga}$$

such that $A_{PL}(X)$ is naturally quasi isomorphic to $C^*(X;\mathbb{Q})$.

A map $f\colon X\to Y$ of spaces is a rational equivalence if it induces an isomorphism $H(X;\mathbb{Q})\to H(Y;\mathbb{Q})$. We say a cdga (A,d) is 1-connected if $H^0(A)=\mathbb{Q}$, $H^1(A)=0$. We say that a space X (resp. a cdga (A,d)) is of finite type if $\dim H^k(X)<\infty$ (resp. $\dim H^k(A)<\infty$). Now A_{PL} induces a bijection

$$\{\text{1-connected finite type top. spaces}\}/_{\mathbb{Q}\text{-equiv.}}\longrightarrow\{\text{1-connected finite type cdgas}\}/_{\text{quasi iso.}}$$

Remark 3.5. This statement can be improved to a statement on the equivalence of certain homotopy categories. The condition on simply connectedness can be weakened as well (although some condition on π_1 remains).

The natural next question is whether we find a preferred kind of model in a given quasi isomorphism type. This leads to the notion of *(Sullivan) minimal models* which we now define.

Let V be a graded vector space. The free graded commutative algebra over V is

$$\Lambda V=\text{Exterior algebra}(V^{odd})\otimes\text{Symmetric algebra}(V^{evem})$$

If $a,b,c,\ldots$ is a homogeneous basis of V we also write $\Lambda(a,b,c,\ldots)$. We use indices to specify degrees unless stated otherwise.

Example 3.6. $\Lambda(a_2,b_3)$ is generated over $\mathbb{Q}$ by

0	1	2	3	4	5	6	7	⋯
1		a	b	a^2	ab	a^3	a^2b	⋯

We consider the following method of specifying a differential:

- Choose a decomposition $V=\bigoplus_{i=0}^{\infty}V_i$ into graded vector spaces (not necessarily the cohomological degree)
- Set $d(V_0)=0$.
- If $(\Lambda V_{\le k},d)$ is a cdga then choose any linear map $d\colon V_{k+1}\to\ker(d|_{\Lambda V_{\le k}})$. It extends uniquely to a derivation d on $\Lambda V_{\le k+1}$ s.t. $d^2=0$.

Definition 3.7. A cdga $(\Lambda V,d)$ as above (with $V=V^{\ge 1}$) is a Sullivan cdga. It is called minimal if additionally $d(V)$ is contained in the set $\Lambda^{\ge 2}V$ of sums of products of at least 2 generators.

Theorem 3.8. *Every cdga (A, d) with $H^0(A) = \mathbb{Q}$ admits a minimal Sullivan model, i.e. there is a quasi isomorphism $(\Lambda V, d) \to (A, d)$ from a minimal Sullivan cdga. The minimal model is unique up to isomorphism*

Definition 3.9. The minimal model of $A_{PL}(X)$ is called the minimal model of X.

Remark 3.10. If X is simply connected then the minimal model $(\Lambda V, d)$ determines X up to rational equivalence. Furthermore one has $V^k \cong \pi_k(X) \otimes \mathbb{Q}$.

Example 3.11.
(i) $\Lambda(x_2, y_{2n+1})$, $dx = 0$, $dy = x^{n+1}$ is given generated over $\mathbb{Q}$ by x^k, $x^k y$, $k \geq 0$ where d vanishes on x^k and $d(x^k y) = x^{k+n+1}$. This is the minimal model of $\mathbb{C}P^n$.

(ii) The model of $\mathbb{C}P^\infty$ is given by Λx with x in degree 2, $dx = 0$. The space BT^r has model $R = \Lambda(x_1, \ldots, x_r)$ where all x_i live in degree 2 and $dx_i = 0$.

(iii) An example of a non-minimal Sullivan cdga is $\Lambda(x, y)$ with $dx = 0$, $dy = x$. It is quasi isomorphic to $\mathbb{Q}$.

Models can be tracked nicely through fibrations where the model of the total space is a twisted tensor product of the models of fiber and base (see [8, Thm 15.3]). Applied to

$$X \to X_T \to BT$$

this specializes to

Theorem 3.12. *Let X be a T-space with minimal model $(\Lambda V, d)$. Then there is a Sullivan model for X_T of the form $(R \otimes \Lambda V, D)$ such that $D|_R = 0$ and for $v \in V$ $D(v) - d(v) \in R^+ \otimes \Lambda V$.*

Conjecture 3.13 (Version 3)**.** *If $(R \otimes \Lambda V, D)$ is a Sullivan cdga with $D|_R = 0$ and finite dimensional cohomology then*

$$\dim H(\Lambda V, d) \geq 2^r$$

where d is the differential induced by D on $(R \otimes \Lambda V)/R^+ \otimes \Lambda V \cong \Lambda V$.

Clearly for the Conjectures we have

$$\text{Version 3} \Rightarrow \text{Version 2} \Rightarrow \text{Version 1}$$

In fact if we restrict to X simply connected in version 1 and 2 and to $V^1 = 0$ in Version 3 then in addition to the above implications we have

$$\text{Version 1} \Rightarrow \text{Version 3}.$$

Proposition 3.14 (cf. [9] Prop. 7.17)**.** *Given $(R \otimes \Lambda V, D)$, $(\Lambda V, d)$ as in the Conjecture and $V^1 = 0$, then we find a simply connected reasonable space with a free T^r-action and model $(\Lambda V, d)$.*

Sketch of proof. There is a space Y with model $(R \otimes \Lambda V, D)$ (this uses $V^1 = 0$). Since $\dim H(Y; \mathbb{Q}) < \infty$ we can take Y to be a finite CW-complex. Let $y_1, \ldots, y_r \in H^2(Y, \mathbb{Z})$ be integral classes lifting the generators of $R = \Lambda(x_1, \ldots, x_r)$ in $H(Y; \mathbb{Q}) \cong H(R \otimes \Lambda V)$.

Now since $BT = K(\mathbb{Z}^r, 2)$ the y_i define a map $f \colon Y' \to BT$ which we use to obtain a pullback T-bundle

$$\begin{array}{ccccc} T & \longrightarrow & X & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ T & \longrightarrow & ET & \longrightarrow & BT \end{array}$$

Studying models of pullback fibrations reveals that the model of X arises from $(R \otimes \Lambda V, D)$ by dividing by the x_i. Hence X has the desired model. □

4 Attacking the TrRC via cdgas

We define $rk_0(X)$ as the maximal r such that T^r acts almost freely on a reasonable space which is rationally equivalent to X. By what we have seen this is computable from the minimal model $(\Lambda V, d)$ of X. It is the maximal r such that one can deform d to a differential D on $\Lambda(x_1, \ldots, x_r) \otimes \Lambda V$ with $|x_i| = 2$, $D(x_i) = 0$ such that $\dim H(\Lambda(x_1, \ldots, x_r) \otimes \Lambda V) < \infty$.

Example 4.1. $X = S^{2n}$, then $\Lambda V = \Lambda(a_{2n}, b_{4n-1})$, $da = 0$, $db = a^2$. What deformations are possible on $\Lambda(x_2, a, b)$? We cannot deform d on a for degree reasons. On b the only choice is $Db = a^2 + \alpha a x^n + \beta x^2 n$ for $\alpha, \beta \in \mathbb{Q}$. But in any case

$$H(\Lambda(x, a, b)) = \mathbb{Q}[x, a]/(a^2 + \alpha a x^n + \beta x^2 n)$$

is infinite dimensional.

The naive takeaway is that one needs enough odd generators (geometrically $\pi_{2n+1}(X) \otimes \mathbb{Q}$) to kill the x_i in cohomology. However the existence of the generators is not sufficient. The following example illustrates another problem: the odd generators killing the x_i can not be "too involved" with the existing differential d.

Example 4.2. The model of the Heisenberg manifold M is $\Lambda(a_1, b_1, c_1)$ with $d(a) = 0 = d(b)$ and $d(c) = ab$. Possible deformations of D on $\Lambda(x, a, b, c)$ are $Da = \lambda_a x$, $Db = \lambda_b x$, and $Dc = ab + \lambda_c x$. However $D^2 = 0$ forces

$$0 = D^2 c = D(\lambda_c x + ab) = Da \cdot b - a \cdot Db = \lambda_a x b - \lambda_b x a$$

Hence $\lambda_a = \lambda_b = 0$. So only c is able to kill added generators in degree 2. In fact setting $\lambda_c = 1$ works and leads to finite dimensional cohomology. But adding more x_i in degree 2 leads to infinite cohomology. Hence $rk_0(M) = 1$.

Remark 4.3. As we have seen in the exercises it is in general not true that the cohomology of a space with a free T^r-action contains an exterior algebra on r-odd generators. Hence it is not true that odd generators that contribute to killing the x_i need to be completely untouched by the differential.

We now try to make these ideas more precise in certain scenarios. We restrict, for the remainder of the section, to simply connected spaces X with $\dim H(X) < \infty$. There is a phenomenon called rational dichotomy (cf. [8, Chapter IV]) stating that a space is either rationally elliptic, i.e.

$$\sum_{k=2}^{\infty} \dim(\pi_k(X) \otimes \mathbb{Q}) < \infty$$

or hyperbolic in which case

$$\sum_{k=2}^{n} \dim(\pi_k(X) \otimes \mathbb{Q})$$

grows exponentially in n.

Example 4.4. (i) Compact homogeneous spaces G/H are elliptic (see exercises).

(ii) Even though many interesting geometric examples elliptic "most" spaces are hyperbolic. Examples are given by $S^2 \vee S^2$ or $\mathbb{C}P^2 \# \mathbb{C}P^2 \# \mathbb{C}P^2$.

Theorem 4.5 ([10])**.** *If $(\Lambda V, d)$ is an elliptic minimal model then* $\dim V^{even} \leq \dim V^{odd}$.

Sketch of proof. Write $V^{even} = P$ and $V^{odd} = Q$. Out of d we build the pure differential d_σ on ΛV with $d(P) = 0$ and splitting $\Lambda V = \Lambda P \oplus \Lambda P \otimes \Lambda^{\geq 1}\Lambda Q$ we define $d_\sigma|_V$ as $d|_V$ projected onto the first factor.

A nontrivial fact which we are going to use without proof is that

$$\dim H(\Lambda V, d) < \infty \iff \dim H(\Lambda V, d_\sigma) < \infty.$$

The !very rough! idea is that when having finitely many generators the polynomial ring ΛP is the only problem for $\dim H(\Lambda V)$ to be finite and that d_σ is just as good at killing ΛP as d is (see [8, Section 32]). Now $(\Lambda V, d_\sigma)$ is a complex

$$0 \leftarrow \Lambda P \leftarrow \Lambda P \otimes Q \leftarrow \Lambda P \otimes \Lambda^2 Q \leftarrow \cdots$$

hence $\dim(\Lambda P/I) < \infty$ where I is the ideal generated by $d_\sigma(Q)$. Writing $\Lambda P = \mathbb{Q}[y_1, \ldots, y_k]$ we find that the radical $\sqrt{I} = (y_1, \ldots, y_k)$ is maximal the height of the Ideal is $\mathrm{ht}(I) = k$. It follows from the Krull height theorem that I can not be generated by less than k elements. Hence $\dim Q \geq k$. □

Definition 4.6. For an elliptic space X we set $\chi(X) = \dim(\pi_{even}(X) \otimes \mathbb{Q}) - \dim(\pi_{odd}(X) \otimes \mathbb{Q})$.

Corollary 4.7 ([2]). *An elliptic space X satisfies $\mathrm{rk}_0(X) \leq -\chi_\pi(X)$.*

Proof. If $(\Lambda V, d)$ is the minimal model of X and $r = \mathrm{rk}_0(X)$ then we find $(\Lambda(x_1, \ldots, x_r) \otimes \Lambda V, D)$ with finite cohomology. Hence

$$\dim V^{even} + r \leq \dim V^{odd}.$$

□

Corollary 4.8. *If X is a product of spheres then $\mathrm{rk}_0(X) = \chi_\pi(X)$ is the number of odd spheres. In particular the TrRC holds.*

Lemma 4.9. *Let $(\Lambda V, d) = (\Lambda(x_{2k} \otimes \Lambda W, d)$ be minimal, $dx = 0$ and consider the induced differential $\overline{d}$ on $\Lambda W \cong \Lambda V/(x)$. Then*

$$\dim H(\Lambda V, d) \geq 2 \dim H(\Lambda W, \overline{d})$$

Sketch of proof. Consider the extension $\Lambda(x, y_{2k-1}) \otimes \Lambda V$ with $dy = dx$. Then the projection

$$(\Lambda(x, y) \otimes \Lambda W, d) \to (\Lambda W, \overline{d})$$

is a quasi isomorphism (this can be shown e.g. via a spectral sequence comparison kind of argument. One calls cdgas of the form $(\Lambda(x, y), d)$ contractible has they are quasi isomorphic to $\mathbb{Q}$). We note that

$$\Lambda(x, y) \otimes \Lambda W = \Lambda V \oplus y \cdot \Lambda V$$

and the cohomology splits as $H(\Lambda V)/(x) \oplus \mathrm{Ann}_{H(\Lambda V)}(x)$. This implies the lemma. □

Corollary 4.10. *The TrRC holds for elliptic spaces with pure minimal models i.e. models of the form $(\Lambda V, d)$ with $d|_{V^{even}} = 0$ and $d(V^{odd}) \subset \Lambda V^{even}$.*

Proof. Let $a = \dim V^{even}$, $b = \dim V^{odd}$. The Quotient $\Lambda V/\Lambda V^{even}$ has trivial differential and Lemma 4.9 yields

$$2^b = \dim H(\Lambda V/\Lambda V^{even}) \Rightarrow \dim H(\Lambda V) \geq 2^{b-a} \geq 2^{\mathrm{rk}_0(\Lambda V, d)}.$$

□

5 The minimal Hirsch-Brown model

Idea: Given a T-space X with minimal model $(\lambda V, d)$, the model $(R, 0) \to (R \otimes \Lambda V, D)$ for $BT \leftarrow X_T$ contains all necessary topological information to prove the TrRC (if it is true). It is in some sense the "best" cdga model. However it is hard to access $H(X) = H(\Lambda V, d)$. An alternative idea is to simplify the model and –although potentially losing important information– making the problem more accessible. We do this by considering $R \otimes \Lambda V$ as a differential graded R-model.

Definition 5.1. A dg R-module is minimal if it is of the form $(R \otimes W, d)$ with $d(W) \subset R^+ \otimes W$.

Proposition 5.2. *(i) Any dg-R-module (M, d) has a minimal model*

$$(R \otimes W, d) \xrightarrow{\text{quasi iso.}} (M, d)$$

unique up to isomorphism.

(ii) The minimal model of $R \otimes \Lambda V$ as above is of the form $(R \otimes H(X), d)$

Very rough sketch. For existence in (i) several different techniques can be used. If cohomology is bounded from below an inductive procedure akin to the one discussed for Sullivan minimal models in the exercises gives a very easy construction. Uniqueness requires some homotopy theory of dg-modules. Another method of construction is to consider the differential $1 \otimes d$ on $R \otimes \Lambda V$ and "compress" the structure into the cohomology $R \otimes H(X)$ (see e.g. [11], [4, Appendix B]). This leads to (ii). □

The model $(R \otimes H(X), d)$ is called the minimal Hirsch-Brown model of the action.

Remark 5.3. (i) It is in general not true that a dg-module over $R = \mathbb{Q}[x_1, \ldots, x_r]$ with $\dim_{\mathbb{Q}} H(R \otimes W) < \infty$ satisfies $\dim W \geq 2^r$ [14]. This would have implied the TrRC!

(ii) $R \otimes H(X)$ is not a cdga but there are traces of multiplicativity left in the form of C_∞-structures. The counterexamples do not carry these and do not disprove the TrRC.

Definition 5.4. A space X is called formal if $A_PL(X)$ is quasi isomorphic to $(H(X), 0)$

A formal space can be viewed as being as simplest kind of space with a given cohomology algebra (from the viewpoint of rational homotopy theory). [5] study actions with certain formality properties where connections to the Buchsbaum-Eisenbud-Horrocks-conjecture arise. In particular the following holds:

Theorem 5.5. *If T^r acts almost freely on a reasonable space X and X_T is formal then* $\dim H(X) \geq 2^r$.

Proof. Let $(R \otimes \Lambda V, D)$ be the model of X_{T^r}. Then by assumption there are quasi isomorphisms

$$(R \otimes H(X), d) \to (R \otimes \Lambda V, D) \to (H(X_T), 0)$$

of dg R-modules where the left hand side is the minimal Hirsch-Brown model. In particular $(R \otimes H(X), d)$ is also the minimal dg R-module model of $(H(X_T), 0)$ which is given by its minimal free resolution. Now by Walkers solution of the total rank conjecture [18] $\dim H(X) \geq 2^r$. □

The question remains what we can say in full generality without additional assumptions. The following discussion is due to Allday and Puppe [4],[16]. For a minimal Hirsch-Brown model $(R\otimes H(X),d)$ of an almost free T^r-action set $F_0=\{x\in H(X)\mid dx=0\}$ and $F_i=\{x\in H(X)\mid d(x)\in R\otimes F_{i-1}\}$. This gives a filtration of $H(X)$ and $R\otimes H(X)$.

For $N\geq 0$ we consider the Koszul complex $K_N=(R\otimes\Lambda(s_1^N,\dots,s_r^N),d)$, where $|s_i^N|=2N+1$ (the exponent N just indicates the degree and is not a power) and $ds_i^N=x_i^{N+1}$

Proposition 5.6. *For $N>>0$ there are dg R-module maps*

$$K_N\xrightarrow{\alpha}(R\otimes H(X),d)\xrightarrow{\beta}K_0$$

mapping $1\mapsto 1$ and $\beta(R\otimes F_i)\subset R\otimes\Lambda^{\leq i}(s_1^0,\dots,s_r^0)$.

Proof. For β choose any map $F_0\to\langle 1_{K_0}\rangle_{\mathbb{Q}}$ with $1\mapsto 1$. If β has been constructed on F_i write $F_{i+1}=F_i\oplus W$ and let $a_1,\dots,a_l$ be a basis of W. Now $d\alpha_j\subset R\otimes F_i$ is closed and so is $\beta(da_j)\subset\Lambda^{\leq i}(s_1^0,\dots,s_1^0)$. Since $H(K_0)=\mathbb{Q}$ we find some $b_j\in\Lambda^{\leq i+1}(s_1^0,\dots,s_r^0)$ with $db_j=a_j$. Now define $\beta(a_j)=b_j$. The construction of α is analogous using that the cohomology in the middle vanishes for sufficiently high degrees. □

Question: What can we say about dg-R-module maps $K_N\xrightarrow{\gamma}K_0$ with $1\mapsto 1$?

Note that $\dim H(X)\geq \mathrm{rk}_R(\gamma)$. In general one does not have $\mathrm{rk}(\gamma)\geq 2^r$ but the technique is still promising. In particular as (ii) below suggests, multiplicative aspects might lead to improved bounds (note that the proofs below do not make use of any multiplicative structure besides the R-module structure!) We write s_{ij}^N for $s_i^N\cdot s_j^N$ etc.

Theorem 5.7. *(i)*

$$\gamma(s_{1\dots r}^N)\notin R\otimes\Lambda^{\leq r-1}(s_1^0,\dots,s_r^0)$$

As a consequence $F_0\subsetneq F_1\subsetneq\dots\subsetneq F_l=H(X)$ has length $l\geq r$.

(ii) If γ is a cdga map, then it is injective.

(iii) γ is injective on

$$R\otimes\langle s_1^N,\dots,s_r^N,s_{123}^N\rangle$$

As a consequence

$$\dim H(X)\geq\begin{cases}2r\\ 2r+1 & \text{if } r\geq 3\end{cases}$$

Partial proof. Assuming (i) in particular $\gamma(f\cdot s_{1\dots r}^N)\neq 0$ for any $f\in R$. But any $a\in K_N$ admits b s.t. $ab=fs_{1\dots r}^N$. Hence if γ is multiplicative it follows that $\gamma(a)\cdot\gamma(b)=\gamma(ab)\neq 0$ proving (ii).

For (iii) given any $\gamma\colon K_N\to K_0$ take $\beta\colon K_N\to K_0$ to be the cdga map with $\beta(s_i)=\gamma(s_i)$ (and extend multiplicatively). Then β is injective by (ii) and hence γ is injective on $R\otimes\langle s_1^N,\dots,s_r^N\rangle$. Adding the extra s_{123}^N requires some more work (see [16]).

Regarding (i) we note that there is only 1 nontrivial homotopy class $K_N\to K_0\simeq\mathbb{Q}$ with $1\mapsto 1$. Consequently γ is homotopic as a dg R-module map to the cdga map $\beta\colon K_N\to K_0$ with $\beta(s_i^N)=x_i^N s_i^0$. Hence γ induces the same map as β on

$$H(K_N/(x_1^{N+1},\dots,x_r^{N+1})\to H(K_0/(x_1^{N+1},\dots,x_r^{N+1}))$$

The class of $s_{1\dots r}^N$ on the left maps to the class of $x_{1\dots r}^N s_{1\dots r}^0$ on the right. Hence $\gamma(s_{1\dots r}^N)$ is given by $x_{1\dots r}^N s_{1\dots r}^0$ up to multiples of x_i^{N+1} and elements in $\mathrm{im}(d)\subset\Lambda^{\leq r-1}(s_1^0,\dots,s_r^0)$ and is thus nonzero in $K_0/(R\otimes\Lambda^{\leq r-1}(s_1^0,\dots,s_r^0))$ □

References

[1] Allday, C. *Transformation Groups: Symplectic Torus Actions and Toric Manifolds.* Hindustan Book Agency, India, 2005.

[2] Allday, C., and Halperin, S. Lie group actions on spaces of finite rank. *Quart J. Math. Oxford 29*, 2 (1978), 69–76.

[3] Allday, C., and Puppe, V. Bounds on the torus rank. *Transformation Groups Poznan* (1986), 1–10.

[4] Allday, C., and Puppe, V. *Cohomological methods in transformation groups*, vol. 32 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1993.

[5] Amann, M., and Zoller, L. The Toral Rank Conjecture and variants of equivariant formality. *J. Math. Pures Appl. 173* (2023), 43–95.

[6] Cairns, G., and Jessup, B. New bounds on the betti numbers of nilpotent Lie algebras. *Comm. Algebra 25*, 2 (1997), 415–430.

[7] Deninger, C., and Singhof, W. On the cohomology of nilpotent Lie algebras. *Bull. Soc. Math. France 116*, 1 (1988), 3–14.

[8] Félix, Y., Halperin, S., and Thomas, J.-C. *Rational homotopy theory*, vol. 205 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2001.

[9] Félix, Y., Oprea, J., and Tanré, D. *Algebraic models in geometry*, vol. 17 of *Oxford Graduate Texts in Mathematics.* Oxford University Press, Oxford, 2008.

[10] Friedlander, J. B., and Halperin, S. An arithmetic characterization of the rational homotopy groups of certain spaces. *Invent. Math. 53*, 2 (1979), 117–133.

[11] Gugenheim, V. On the chain complex of a fibration. *Illinois J. Math. 16*, 3 (1972), 398–414.

[12] Halperin, S. Rational homotopy and torus actions. In *Aspects of topology*, vol. 93 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 1985, pp. 293–306.

[13] Hsiang, W.-y. *Cohomology theory of topological transformation groups.* Springer-Verlag, New York, 1975. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 85.

[14] Iyengar, S. B., and Walker, M. E. Examples of finite free complexes of small rank and small homology. *Acta Math. 221*, 1 (2018), 143–158.

[15] Jessup, B., and Lupton, G. Free torus actions and two-stage spaces. *Math. Proc. Cambridge Philos. Soc. 137*, 1 (2004), 191–207.

[16] Puppe, V. Multiplicative aspects of the Halperin-Carlsson conjecture. *Georgian Math. J. 16*, 2 (2009), 369–379.

[17] Sullivan, D. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, 47 (1977), 269–331 (1978).

[18] Walker, M. E. Total Betti numbers of modules of finite projective dimension. *Ann. of Math. (2) 186*, 2 (2017), 641–646.