Lecture Notes on Free $(\mathbb{Z}/p)^r$-Actions on Products of Spheres

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Abstract

These are the extended lecture notes for the masterclass on "Rank Conjectures Across Algebra and Topology", which will take place in June 24-28, 2024 at the University of Copenhagen. This is a survey/summary of known results on the topic from various published articles and books cited in the text. No originality is claimed. Last section is based on the joint work with Okutan.

Contents

1 Rank Conjectures for Free Actions: A Survey

Let G be a finite group and X be a CW-complex with a G-action on it. We say the action is **cellular** if G acts by permuting the cells in X. We also assume that G acts on X in such a way that if an element $g \in G$ fixes a cell, then it fixes it pointwise. The action is said to be **free** if for every element $x \in X$, the isotropy subgroup $G_x = \{g \in G \,|\, gx = x\}$ is the trivial subgroup. In 1944, Smith proved the following theorem:

Theorem 1 (Smith [30]). *Let G be a finite group and p be a prime number. If G acts freely and cellularly on a finite-dimensional CW-complex X which has the mod-p homology of an n-sphere, then G does not include $\mathbb{Z}/p \times \mathbb{Z}/p$ as a subgroup.*

Smith proved this result by showing that if G admits such an action then the mod-p group cohomology of G is periodic, i.e., there is a $d \geq 1$ such that $H^n(G; \mathbb{F}_p) \cong H^{n+d}(G; \mathbb{F}_p)$ for all $n \geq 0$. An easy argument in group cohomology shows that then G does not include $\mathbb{Z}/p \times \mathbb{Z}/p$ as a subgroup. We discuss Smith's theorem in detail in Section 2.4.

In 1957 Conner [13] extended Smith's result to free actions on a product of two equal dimensional spheres $S^n \times S^n$ using the associated Borel fibration (see Section 3.1 for a definition). He proved that if G acts freely on a finite-dimensional CW-complex X which has mod-p homology of $S^n \times S^n$, then G does not include $\mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p$ as a subgroup. Later Heller [16] extended Conner's result further to a product of two spheres $S^n \times S^m$ with arbitrary dimensions. In his proof Heller uses Tate cohomology and assumes only that X has mod-p homology of $S^n \times S^m$.

Motivated by Smith, Conner, and Heller's results, the following conjecture has been made:

Conjecture 2 (Rank Conjecture for $(\mathbb{Z}/p)^r$-actions). *If $G = (\mathbb{Z}/p)^r$ acts freely on $X = S^{n_1} \times \cdots \times S^{n_k}$, then $r \leq k$.*

In the above statement X is a topological space homeomorphic to $S^{n_1} \times \cdots \times S^{n_k}$, and G acts on X with a continuous map $G \times X \to X$. The assumption on X can be replaced by other assumptions depending on what kind of tools we want to use. For example, one can assume that X is a smooth manifold diffeomorphic to $S^{n_1} \times \cdots \times S^{n_k}$ with smooth G-action, or X is a $\mathbb{Z}_{(p)}$-homological manifold. In the light of above results the expectation is that the conjecture may hold under much weaker homological assumptions on X. In the known results on rank conjecture, one generally assume X is a finite or finite-dimensional G-CW-complex such that

1. X is homotopy equivalent to $S^{n_1} \times \cdots \times S^{n_k}$, or
2. $H^*(X; R) \cong H^*(S^{n_1} \times \cdots \times S^{n_k}; \mathbb{F}_p)$ as Steenrod algebras, or
3. $H^*(X; R) \cong H^*(S^{n_1} \times \cdots \times S^{n_k}; R)$ as R-algebras where $R = \mathbb{Z}_{(p)}$ or $\mathbb{F}_p$, or
4. $H_*(X; R) \cong H_*(S^{n_1} \times \cdots \times S^{n_k}; R)$ as R-modules where $R = \mathbb{Z}_{(p)}$ or $\mathbb{F}_p$.

Therefore one can consider many different versions of the rank conjecture depending on the assumptions on X. It would be interesting to see the proofs or counterexamples to any of these versions. It is also interesting to ask whether or not any two versions of the topological rank conjecture is equivalent to each other.

Rank conjecture is a special case of the following conjecture due to Carlsson [12].

Conjecture 3 (Carlsson's conjecture)**.** *If $G = (\mathbb{Z}/p)^r$ acts freely on a finite, connected CW-complex X, then*

$$2^r \leq \sum_{i=0}^{\dim X} \dim_{\mathbb{F}_p} H_i(X; \mathbb{F}_p).$$

Note that the total homology dimension of a product of k spheres is equal to 2^k, hence Conjecture 2 follows from Conjecture 3. There is also a rational version of Carlson's conjecture for $G = (S^1)^r$ actions due to Halperin [18, Problem 1.4]. It says that if $G = (S^1)^r$ and X is a finite, free G-CW-complex, then

$$2^r \leq \sum_{i=0}^{\dim X} \mathrm{rk}_{\mathbb{Q}} H_i(X; \mathbb{Q}).$$

It is known that Halperin's conjecture is true for $X = S^{n_1} \times \cdots \times S^{n_k}$ (see [20]). Carlson's and Halperin's conjectures are still open in general.

There is an algebraic version of Carlsson's conjecture which states that if $G = (\mathbb{Z}/p)^r$ and C_* is a finite chain complex of free $\mathbb{F}_p G$-modules with nonzero homology, then $\dim_{\mathbb{F}_p} H_*(C_*) \geq 2^r$. Recently, Iyenger and Walker [22] gave a counterexample to the algebraic version of the rank conjecture for $p \geq 3$ and $r \geq 8$. Ruping and Stephan [27] proved that the counterexample complexes constructed by Iyenger and Walker can not be realized topologically. So Conjecture 2 is still open in general.

The main purpose of these notes is to discuss some of the known positive results for the special cases of Conjecture 2. The first positive result for Conjecture 2 are due to Carlsson that proves the conjecture for the case $n_1 = \cdots = n_k$ under the assumption that the induced action on the mod-p homology of X is trivial.

Theorem 4 (Carlsson [10], [11])**.** *Suppose $G = (\mathbb{Z}/p)^r$ acts freely on a finite complex X, where X is homotopy equivalent to $(S^n)^k$, and suppose that G acts trivially on $H^*(X; \mathbb{F}_p)$. Then $r \leq k$.*

Carlsson proves this theorem first for $p = 2$ in [10] using the Serre spectral sequence for the Borel fibration $X \to EG \times_G X \to BG$. The main ingredients of the proof are the product structure of the Serre spectral sequence and a version of Kudo's trangression theorem for Steenrod operations. In Section 3, we discuss the proof given in [10] for the $p = 2$ case of Theorem 4.

In 1982, Carlsson [11] proves the p odd version of Theorem 4 by constructing a specific model for $C_*(X)$ and by using some results from commutative algebra. Later Browder [6], and Benson and Carlson [3] also gave proofs for Theorem 4 using different methods. Browder used Tate cohomology and exponents of cohomology groups, and Benson and Carlson used the L_ζ-modules and support varieties of modules. We discuss Browder's proof in Section 4.

In 1988, Adem and Browder [2] proved that Conjecture 2 holds for free actions of $G = (\mathbb{Z}/p)^r$ on a product of equidimensional spheres $(S^n)^r$ without the assumption that the action on homology is trivial for all p and n except when $p = 2$ and $n = 1, 3, 7$.

Theorem 5 (Adem-Browder [2])**.** *Let $G = (\mathbb{Z}/p)^r$ act freely on $X = (S^n)^k$. Assume that $n \neq 1, 3, 7$ when $p = 2$. Then $r \leq k$.*

The case $p = 2$ and $n = 1$ is resolved later in [32]. The methods used in [32] are very different than the methods used for the general proof. In [32], we use group extension theory and special extension classes that were also used in the classification of Bieberbach groups. The cases $p = 2$ and $n = 3, 7$ are still open.

The proof of Theorem 5 is written under the assumption that X is an orientable $\mathbb{Z}_{(p)}$-homology manifold, but it can be replaced by an argument that uses only the assumption that X is a finite complex with mod-p homology of $(S^n)^k$. The main ingredient of the proof is an inequality proved for $\mathbb{Z}_{(p)}G$-lattices for $G = (\mathbb{Z}/p)^r$. Using this inequality Adem and Browder [2] proves the following sharper result for the rank of elementary abelian p-subgroups acting freely on $X = (S^n)^r$.

Theorem 6 (Adem-Browder [2, Thm 4.1])**.** *Let $G = (\mathbb{Z}/p)^r$, p odd, act freely on an orientable $\mathbb{Z}_{(p)}$-homology manifold X with $H^*(X; \mathbb{Z}_{(p)}) \cong H^*((S^n)^k; \mathbb{Z}_{(p)})$. Then*

$$r \leq \dim_{\mathbb{F}_p} H_n(X; \mathbb{F}_p)^G + \Big(\frac{1}{p-1}\Big)(\dim H_n(X; \mathbb{F}_p) - \dim H_n(X; \mathbb{F}_p)^G).$$

One of the consequences of this stronger inequality is that if p odd, and $G = (\mathbb{Z}/p)^r$ acts freely on $X = (S^n)^r$, then G must act trivially on the mod-p homology of X.

For free actions on an arbitrary products of spheres, there are two other results that we would like to mention. The first one is due to Hanke for the case where p is large compared to the dimension of the space.

Theorem 7 (Hanke [19])**.** *Let $X = S^{n_1} \times \cdots \times S^{n_k}$ and k_0 denote the number of odd dimensional spheres in X. If $p > 3 \dim X$ and $(\mathbb{Z}/p)^r$ acts freely on X, then $r \leq k_0$.*

This theorem suggests that for odd primes, the upper bound in Conjecture 2 should be replaced by k_0, the number of odd dimensional spheres in X. Hanke uses the tame homotopy theory in his proof. Since the techniques of the proof are out of the scope of these notes, we will not explain Hanke's proof in these notes. Another result for free actions on products of spheres with different dimensions is the following:

Theorem 8 (Okutan-Yalçın [25])**.** *Suppose $G = (\mathbb{Z}/p)^r$ for a prime p and k, l are positive integers. Then there is an integer N that depends only on k, l, and G, such that if G acts freely and cellularly on a finite-dimensional CW-complex X homotopy equivalent to $S^{n_1} \times \cdots \times S^{n_k}$ where $n_i \geq N$ and $|n_i - n_j| \leq l$ for all i, j, then $r \leq k$.*

This theorem says that if $X \simeq S^{n_1} \times \cdots \times S^{n_k}$ where the average dimension of spheres is large compared to the differences $|n_i - n_j|$ of dimensions, then the rank conjecture is true for free actions on X. The main ingredients for the proof of this theorem is a theorem of Habegger [17] and an observation about cohomology of finite groups due to Pakianathan [26]. We explain the proof of Theorem 8 in detail in Section 5.

2 Homological Methods for Studying Group Actions

2.1 Group cohomology

Let G be a finite group, and k be a commutative ring. The group ring kG is the ring whose elements are the formal sums $\sum_{g\in G} a_g g$ where $a_g \in k$, where the multiplication is given by

$$\Big(\sum_{g\in G} a_g g\Big)\Big(\sum_{g\in G} b_g g\Big) = \sum_{g\in G}\Big(\sum_{hk=g} a_h b_k\Big) g.$$

We often consider the cases where $k = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_{(p)}$, or where k is a field of characteristic p.

Definition 9. A k-module M together with a G-action satisfying the following linearity conditions is called a **G-module**:
(1) For every $g \in G$, $m, m' \in M$, $g(m + m') = gm + gm'$;
(2) For every $g \in G$, $m \in M$, and $\lambda \in k$, $g(\lambda m) = \lambda(gm)$.

Every kG-module M can be considered as a G-module by taking its underlying k-module with the G-action defined by multiplication with $1g \in \mathbb{Z}G$. Conversely, every G-module can be considered as a kG-module by linearly extending the action of G on M to an action of the ring kG.

The ring k is a one-dimensional k-module. We can consider it as a kG-module with the trivial G-action $g\lambda = \lambda$. We refer to this module as the **trivial kG-module k**. A **free kG-module resolution** of the trivial module k is an exact sequence of the form

$$(F_*, \epsilon) : \cdots \to F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \to \cdots \to F_2 \xrightarrow{\partial_1} F_1 \xrightarrow{\partial_0} F_0 \xrightarrow{\varepsilon} k \to 0$$

where each F_i is a free kG-module. Given a kG-module M, consider the cochain complex $\mathrm{Hom}_{kG}(F_*, M)$ defined by

$$0 \to \mathrm{Hom}_{kG}(F_0, M) \xrightarrow{\delta^0} \mathrm{Hom}_{kG}(F_1, M) \xrightarrow{\delta^1} \cdots \to \mathrm{Hom}_{kG}(F_n, M) \xrightarrow{\delta^n} \mathrm{Hom}_{kG}(F_{n+1}, M) \to \cdots$$

where $\delta^n(f) = (-1)^{n+1} f \circ \partial_{n+1}$ for every $f \in \mathrm{Hom}_{kG}(F_n, M)$.

Definition 10. The **cohomology of G with coefficients in M** is defined by

$$H^n(G; M) := H^n(\mathrm{Hom}_{kG}(F_*, M), \delta^*) = \ker \delta^n / \mathrm{im}\, \delta^{n-1}.$$

By the comparison theorem in homological algebra, projective/free resolution of k is unique up to chain homotopy. So the definition of group cohomology does not depend on the free resolution F_*.

Exercise 11. Using the definition of group cohomology, prove that for every kG-module M, $H^0(G; M) \cong M^G := \{m \in M \mid gm = m \text{ for all } g \in G\}$.

Example 12. Suppose that p is a prime number, and $G = C_p = \langle g \mid g^p = 1\rangle$ is the cyclic group of order p. Let k be a field with characteristic p. Then there is a free kG-resolution of k of the form

$$(F_*, \varepsilon) : \ldots \longrightarrow kG \xrightarrow{N_G} kG \xrightarrow{1-g} kG \xrightarrow{N_G} kG \xrightarrow{1-g} kG \xrightarrow{\varepsilon} k \longrightarrow 0$$

where $N_G = 1+g+\cdots+g^{p-1}$ is the norm element, and ε is the augmentation map defined by $\varepsilon(\sum_g a_g g) = \sum_g a_g$. Applying the Hom-functor $\mathrm{Hom}_{kG}(-;k)$, we obtain a cochain complex of the form

$$0 \longrightarrow k \xrightarrow{0} k \xrightarrow{0} k \xrightarrow{0} \cdots.$$

Hence for every $n \geq 0$, we have $H^n(C_p;k) \cong k$ as k-vector spaces.

Exercise 13. Verify that the chain complex in Example 12 is a resolution for k as a kG-module.

For every subgroup $H \leq G$, there is a homomorphism $\mathrm{Res}^G_H : H^*(G;M) \to H^*(H;\mathrm{Res}^G_H M)$ induced by the chain map

$$\mathrm{Res}^G_H : \mathrm{Hom}_{kG}(F_*, M) \to \mathrm{Hom}_{kH}(\mathrm{Res}^G_H F_*, \mathrm{Res}^G_H M)$$

defined by the restriction of kG-homomorphisms to kH-homomorphisms. For every subgroup $H \leq G$ with $|G:H| < \infty$, there is a chain map in the other direction

$$\mathrm{Tr}^G_H : \mathrm{Hom}_{kH}(\mathrm{Res}^G_H F_*, \mathrm{Res}^G_H M) \to \mathrm{Hom}_{kG}(F_*, M)$$

defined by $\mathrm{Tr}^G_H(f) = \sum_{gH \in G/H} gfg^{-1}$ which induces a homomorphism

$$\mathrm{Tr}^G_H : H^*(H;\mathrm{Res}^G_H M) \to H^*(G;M).$$

Using the definition it is easy to see that for every $H \leq G$ with finite index, $\mathrm{Tr}^G_H \mathrm{Res}^G_H u = |G:H| u$ for every $u \in H^n(G;M)$. This shows in particular that for every finite group G, we have $|G| \cdot H^n(G;M) = 0$ for $n \geq 0$. If G is a finite group, then $H^n(G;\mathbb{Z})$ is a finite abelian group whose exponent divides the order of the group.

Definition 14. The tensor product $C_* \otimes D_*$ of two chain complexes C_*, D_* is a chain complex with $(C_* \otimes D_*)_n = \bigoplus_{i+j=n} C_i \oplus D_j$ with the boundary map defined by

$$\partial_n(x \otimes y) = \partial_i(x) \otimes y + (-1)^i x \otimes \partial_j(y)$$

for $x \in C_i$ and $y \in D_j$.

Given a free resolution $F_* \xrightarrow{\varepsilon} k$, consider the tensor product $F_* \otimes F_*$ together with the augmentation map $F_* \otimes F_* \xrightarrow{\varepsilon} k$ defined by the homomorphism $F_0 \otimes F_0 \xrightarrow{\varepsilon\otimes\varepsilon} k \otimes k \xrightarrow{\mu} k$, where μ is the multiplication map. The augmented chain complex $F_* \otimes F_* \to k$ is a free kG-resolution of the trivial module k. By comparison theorem in homological algebra, there is a chain map $\Delta_* : F_* \to F_* \otimes F_*$ which induces the identity map on k. Any such chain map is called a **diagonal approximation**.

Using a diagonal approximation $\Delta_* : F_* \to F_* \otimes F_*$, one can define cup product on $H^*(G;k) := \bigoplus_{i \geq 0} H^i(G;k)$ using the composition

$$\mathrm{Hom}_{kG}(F_*,k) \otimes_k \mathrm{Hom}_{kG}(F_*,k) \xrightarrow{\psi} \mathrm{Hom}_{kG}(F_* \otimes_k F_*, k \otimes k) \xrightarrow{\Delta^*} \mathrm{Hom}_{kG}(F_*, k \otimes k) \xrightarrow{\mu_*} \mathrm{Hom}_{kG}(F_*, k)$$

where the first map ψ is defined by

$$\psi(f_1 \otimes f_2)(x_1 \otimes x_2) = (-1)^{\deg f_2 \cdot \deg x_1} f_1(x_1) \otimes f_2(x_2),$$

the second map is defined by $\Delta^*(f)(x) = f(\Delta(x))$, and the last map is induced by multiplication map $\mu : k \otimes k \to k$. These chain maps induces a homomorphism of cohomology modules

$$H^n(\Delta^* \circ \psi) : H^n(\mathrm{Hom}_{kG}(F_*, k) \otimes_k \mathrm{Hom}_{kG}(F_*, k)) \to H^n(\mathrm{Hom}_{kG}(F_*, k)).$$

Recall that by standard properties of tensor product of chain complexes, we have an homomorphism

$$\theta : H^p(\mathrm{Hom}_{kG}(F_*, k)) \otimes_k H^q(\mathrm{Hom}_{kG}(F_*, k)) \to H^{p+q}(\mathrm{Hom}_{kG}(F_*, k) \otimes_k \mathrm{Hom}_{kG}(F_*, k))$$

defined by $[f_1] \otimes [f_2] \to [f_1 \otimes f_2]$.

Definition 15. For every $p, q \geq 0$, the **cup product**

$$\cup : H^p(G; k) \otimes_k H^q(G; k) \to H^{p+q}(G, k)$$

is defined to be the composition $\theta \circ H^n(\Delta^* \circ \psi)$.

To shorten the notation we write uv for the cup product $u \cup v$. With cup product as the ring multiplication, $H^*(G; k)$ is a graded commutative k-algebra with identity $1 \in H^0(G; k) \cong k$. We have the following computation for the cohomology algebra of the cyclic group of order p.

Proposition 16 ([8, Prop 4.5.1])**.** *Let $G = C_p$, p a prime number, and k be a field with characteristic p. Then there is an isomorphism of k-algebras*

$$H^*(C_p; k) \cong \begin{cases} k[x] & \text{if } p = 2 \\ \wedge_k(y) \otimes k[x] & \text{if } p > 2 \end{cases}$$

where $|x| = 1$ if $p = 2$, and $|y| = 1$, $|x| = 2$ if $p > 2$.

Exercise 17 (Brown [7, pp 108])**.** Suppose that $G = \langle g \mid g^p = 1 \rangle \cong C_p$ is a finite cyclic group of order p, and k is a field of char p. Let $F_* \to k$ be the periodic free kC_p-resolution of k defined in Example 12. Then show that the map $\Delta : F_* \to F_* \otimes_k F_*$ defined by

$$\Delta_{m,n}(1) = \begin{cases} 1 \otimes 1 & \text{if } m \text{ is even} \\ 1 \otimes g & \text{if } m \text{ is odd}, n \text{ is even} \\ \sum_{0 \leq i < j \leq n-1} g^i \otimes g^j & \text{if } m \text{ is odd}, n \text{ is odd} \end{cases}$$

is a diagonal approximation. Using this diagonal approximation, prove that the isomorphism in Proposition 16 holds.

If H and K are two groups, then $kH \otimes_k kG \cong k[H \times K]$ is a free $H \times K$-module. Using this, one can show that if $F_* \to k$ is a free kH-resolution and $F'_* \to k$ is a free kK-resolution, then $F \otimes_k F'$ is a chain complex of free $k[H \times K]$-modules. By Künneth theorem, we obtain that $F_* \otimes_k F'_* \to k$ is a free $k[H \times K]$-resolution of k. Applying the Hom-functor $\mathrm{Hom}_{k[H \times K]}(-, k)$, and using the Künneth theorem again, we obtain that there is an isomorphism

$$H^*(H \times K; k) \cong H^*(H; k) \otimes_k H^*(K; k).$$

With some extra work, one can show that this is an isomorphism of k-algebras [8, Prop 4.3.5]. As a consequence we obtain the following calculation.

Proposition 18 ([8, Prop 4.5.4])**.** *Let k be a field with characteristic p. Then*

$$H^*((C_p)^r;k) \cong \begin{cases} k[x_1,\dots,x_r] & \text{if } p=2 \\ \wedge_k(y_1,\dots,y_r)\otimes k[x_1,\dots,x_r] & \text{if } p>2 \end{cases}$$

where $\deg x_i = 1$ *if* $p=2$, *and* $\deg y_i = 1$, $\deg x_i = 2$ *if* $p>2$.

Exercise 19. Give a full proof for Proposition 18.

2.2 G-simplicial complexes and their homology

Let G be a finite group. A G**-simplicial complex** X is a simplicial complex with a G-action on its set of vertices in such a way that for every simplex $\sigma = \{v_0,\dots,v_n\}$ in X, $g\sigma = \{gv_0,\dots,gv_n\}$ is a simplex in X. The realization $|X|$ of a G-simplicial complex X is a topological space with a continuous G-action.

A G-simplicial complex X is **admissible** if whenever an element $g\in G$ fixes a simplex $\sigma\in X$, then it fixes all its vertices. If X is an admissible G-complex, then for every $H\le G$,

$$X^H = \{\sigma\in X \mid h\sigma=\sigma \text{ for all } h\in H\}$$

is a subcomplex of X. In this case we have $|X|^H \cong |X^H|$.

An admisible G-simplical complex is **regular** if it satisfies the following condition: If $g_0,g_1,\dots,g_n\in G$ and $\{v_0,\dots,v_n\}$ and $\{g_0v_0,\dots,g_nv_n\}$ are both simplices in X, then there exists an element g in G such that $gv_i = g_iv_i$ for all i. If X is regular, then the set of orbits $[\sigma]_G = \{g\sigma \mid g\in G\}$ of simplices in X forms a simplicial complex X/G such that $|X|/G \cong |X/G|$.

Exercise 20. Verify the above statements for fixed points and orbit spaces. Prove that if X is a G-simplicial complex, then $sd(X)$ is admissible. Also show that if X is admissible, then $sd(X)$ is regular.

Let k be a commutative ring. The (oriented) simplicial chain complex $C_*(X;k)$ of a simplicial complex X with coefficients in k is defined as follows: For each simplex σ in X, choose a total ordering for the vertices of σ and denote these oriented simplices by $[v_0,\dots,v_n]$ where $v_0<\cdots<v_n$. The n-th chain module $C_n(X;k)$ is the free k-module with basis given by all oriented simplices $\{v_0,\dots,v_n\}$ of X. For the tuples of vertices ordered in a different way, we have the following identification: If σ is a permutation of $\{0,\dots,n\}$, then

$$[v_{\sigma(0)},\dots,v_{\sigma(k)}] = \mathrm{sign}(\sigma)[v_0,\dots,v_k].$$

For each $k\ge 1$, the boundary map $\partial : C_n(X;k)\to C_{n-1}(X;k)$ is defined by

$$\partial_n[v_0,\dots,v_n] = \sum_{i=0}^{n}(-1)^i[v_0,\dots,\widehat{v}_i,\dots,v_k].$$

Exercise 21. Let X be the 1-dimensional simplicial complex whose vertex set is $V = \{a,b,c,d\}$ and whose 1-simplices are given by $\{a,b\}$, $\{b,c\}$, $\{c,d\}$, and $\{d,a\}$. Choose an orientation for each simplex and write the simplicial chain complex of X. Show that $H_1(X;k)\cong H_0(X;k)\cong k$.

If X is a G-simplicial complex, then G permutes the simplices of X. Each chain module $C_n(X; k)$ is a kG-module and boundary maps are kG-module homomorphisms. In general $C_n(X; k)$ does not have to be permutation module, since the $g \in G$ action may change the sign of a basis element $[v_0, \dots, v_n]$. If X is admissible, this can not happen anymore: the stabilizer of each simplex $\sigma = \{v_0. \dots, v_n\}$ acts on the tuple $[v_0, \dots, v_n]$ trivially. In this case the chain module $C_n(X; k)$ is a permutation kG-module. Furthermore, if the G-action on X is free, then the chain complex $C_*(X; k)$ is a chain complex of free kG-modules.

Example 22. Let $G = \langle g \,|\, g^2 = 1 \rangle \cong C_2$. Consider the 1-dimensional simplicial complex X whose vertex set is $V = \{a, b, c, d\}$ and whose 1-simplices are given by $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, and $\{d, a\}$. Let $g \in G$ act on V by $ga = c$ and $gb = d$. It is easy to see that with this action, X is an admissible G-simplicial complex. The realization of X is a square with corners given by vertices of X and edges are given by the 1-simplices. Note that $|X|$ is G-homeomorphic to G-space S^1 with the antipodal action defined by $gx = -x$. The simplicial chain complex of X is of the form

$$0 \to C_1(X; k) \xrightarrow{\partial_1} C_0(X; k) \to 0$$

where

$$C_1(X; G) \cong k[a, b] \oplus k[b, c] \oplus k[c, d] \oplus k[d, a] \cong kG \cdot [a, b] \oplus kG \cdot [b, c]$$

and

$$C_0(X; G) \cong k[a] \oplus k[b] \oplus k[c] \oplus k[d] \cong kG \cdot [a] \oplus kG \cdot [b].$$

The boundary map is defined by $\partial_1([a, b]) = [b] - [a]$ and $\partial_1([b, c] = [c] - [b]$. The homology of this chain complex is $H_1(X; k) \cong H_0(X; k) \cong k$. This gives an exact sequence of kG-modules

$$0 \to k \to kG \oplus kG \to kG \oplus kG \to k \to 0.$$

Splicing these short exact sequences together one obtains a periodic free kG-resolution of k.

Example 23. Let $G = \langle g_1, g_2 \,|\, g_1^2 = g_2^2 = (g_1 g_2)^2 = 1 \rangle \cong C_2 \times C_2$. Consider the 1-dimensional simplicial complex X given above with G-action defined by $g_1 a = c$, $g_1 b = b$, $g_2 a = a$, and $g_2 b = d$. Then X is an admissible G-simplicial complex and the chain complex $C_*(X; k)$ is of the form

$$0 \to kG[a, b] \xrightarrow{\partial_1} k[G/\langle g_2 \rangle] \cdot [a] \oplus k[G/\langle g_1 \rangle] \cdot [b] \to 0$$

where boundary map sends the basis element $[a, b]$ to $[b] - [a]$. This gives an exact sequence of kG-modules

$$0 \to k \to kG \to k[G/\langle g_2 \rangle] \oplus k[G/\langle g_1 \rangle] \to k \to 0.$$

Note that in this case the chain modules are not free kG-modules, so the sequence above does not give a periodic free kG-resolution of k.

Exercise 24. Consider the action of the symmetric group $G = S_3$ on the simplicial complex X with vertex set $V = \{1, 2, 3\}$ and edge set $S = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ given by the permutation of the vertices with the natural action. The realization of X is the boundary of a triangle with corners labeled as 1, 2, 3. Observe that this action is not admissible but the G-action on the barycentric subdivision $Y = sd(X)$ is admissible. Write the chain modules and boundary maps for the simplicial chain complex of X and Y.

2.3 G-CW-complexes

Another convenient category to study the G-spaces is the category of G-CW-complexes.

Definition 25. A G-space is a G**-CW-complex** if there is a filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-1} \subseteq X_n \subseteq \cdots$$

of X such that

1. For $n \geq 0$, X_n is obtained form X_{n-1} by attaching G-orbits of cells, i.e. there is a G-pushout diagram

$$\begin{tikzcd} \coprod_{i\in I_n} G/H_i \times S^{n-1} \arrow[r, "\coprod f_i^n"] \arrow[d, hook] & X_{n-1} \arrow[d, hook] \\ \coprod_{i\in I_n} G/H_i \times D^n \arrow[r, "\coprod h_i^n"] & X_n \end{tikzcd}$$

2. X has weak topology with respect to the filtration $\{X_n\}$, i.e. $B \subseteq X$ is closed if and only if $B \cap X_n$ is closed for all $n \geq -1$.

For each $i \in I_n$, the subspace $e_i^n = h_i^n(G/H_i \times \mathrm{int}D^n)$ is called an orbit of open n-cells of X.

Let G be a finite group. If G acts on a CW-complex X by permuting its cells is called a **cellular action**. It is also convenient to assume that the G-action satisfies the property that if an element $g \in G$ fixes a cell, then it fixes it pointwise. Note that if X is a G-CW-complex then G acts on X cellularly and it satisfies the additional fixed point property. Conversely, If X is a CW-complex and G-acts cellularly on X, satisfying the additional fixed point property, then X has a G-CW-complex structure.

The following facts are well-known for G-CW-complexes for a finite group G (see [23]):

1. If X is an admissible G-simplicial complex then its realization is a G-CW-complex. By an equivariant version of simplicial approximation theorem, every G-CW-complex is G-homotopy equivalent to a simplicial G-complex.

2. Every closed smooth G-manifold admits a G-CW-complex structure.

3. A G-simplicial set is defined as a simplicial object in G-sets. The realization of a G-simplicial set is a G-CW-complex.

4. Product of two G-CW-complexes is a G-CW-complex if we take $X \times Y$ with compactly generated topology. If one of the complexes X or Y is a finite complex, then this is true also with product topology.

The cellular chain complex $C_*(X; k)$ of a G-CW-complex is a chain complex where $C_n(X; k)$ is a free k-module generated by n-dimensional cells. From the definition of G-CW-complexes, we have

$$C_n(X; k) \cong \bigoplus_{i\in I_n} k[G/H_i]$$

as kG-modules. The boundary map $\partial_n : C_n(X; k) \to C_{n-1}(X; k)$ is a kG-module homomorphism. If G-action on X is free, then $C_*(X; k)$ is a chain complex of free kG-modules.

2.4 Proof of Smith's theorem

Before we prove Smith's theorem we make a few observations on cohomology of groups. Let G be a finite group and k be a field of characteristic p. Then the group ring kG is isomorphic to its dual $kG^* = \mathrm{Hom}_k(kG, k)$. Using this we can show the following:

Lemma 26. *If F is a free kG-module, then $H^n(G; F) = 0$ for $n \geq 1$.*

Proof. Let F_* be a free kG-resolution of k, and let $\widetilde{F}_* : \cdots \to F_1 \to F_0 \to k \to 0$ denote the corresponding augmented complex. We have

$$\mathrm{Hom}_{kG}(\widetilde{F}_*, kG) \cong \mathrm{Hom}_{kG}(\widetilde{F}_*, \mathrm{Hom}_k(kG, k)) \cong \mathrm{Hom}_{kG}(\widetilde{F}_* \otimes_k kG, k) \cong \mathrm{Hom}_k(\widetilde{F}_*, k).$$

Hence the cochain complex $\mathrm{Hom}_{kG}(\widetilde{F}_*, kG)$ is an exact sequence since $\mathrm{Hom}_k(\widetilde{F}_*, k)$ is exact. This gives that $H^n(G; kG) = 0$ for $n \geq 1$. Hence for every free kG-module $F \cong \oplus_{i\in I} \mathbb{Z}G$, we have $H^n(G; F) \cong \oplus_{i\in I} H^n(G; \mathbb{Z}G) = 0$ for $n \geq 1$. $\square$

Using this vanishing result and the long exact sequences for group cohomology, we obtain the following:

Lemma 27. *Let $0 \to A \to F \to B \to 0$ be a short exact sequence of kG-modules where F is a free kG-module. Then $H^i(G; B) \cong H^{i+1}(G; A)$ for $i \geq 1$.*

Exercise 28. Show that if $0 \to A \to F_{k-1} \to \cdots \to F_0 \to B \to 0$ is an exact sequence of kG-modules where $F_0, \dots, F_{k-1}$ are free kG-modules, then $H^i(G; B) \cong H^{i+k}(G; A)$ for all $i \geq 1$.

We now give the proof of Smith's theorem stated in Section 1.

Proof of Theorem 1. Let G be a finite group and p be a prime number. Suppose that X is a finite-dimensional free G-CW-complex X such that $H_*(X; \mathbb{F}_p) \cong H_*(S^n; \mathbb{F}_p)$, $n \geq 1$. We claim that then G does not include $C_p \times C_p$ as a subgroup. Assume contrary that G has a subgroup $P \leq G$ such that $P \cong C_p \times C_p$.

Consider the P-action on X via restriction. Since this action is also free, the chain complex of X, with coefficients in a field k of characteristic p,

$$C_*(X; k) : 0 \to C_d \to C_{d-1} \to \cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_0 \to k \to 0t$$

is a chain complex of free kP-modules. We have $H_n(X; k) = Z_n/B_n \cong k$ where $Z_n = \ker \partial_n$ and $B_n = \mathrm{im}\, \partial_{n+1}$. This gives two exact sequences

$$0 \to C_d \to C_{d-1} \to \cdots \to C_{n+1} \to C_n \xrightarrow{q} C_n/B_n \to 0 \tag{1}$$

and

$$0 \to k \to C_n/B_n \xrightarrow{\overline{\partial}_n} C_{n-1} \to C_{n-2} \to \cdots \to C_0 \to k \to 0. \tag{2}$$

where q is the quotient map and $\overline{\partial}_n$ is the map induced by ∂_n by taking quotient with $B_n \subseteq Z_n = \ker \partial_n$.

The last two terms of the exact sequence in (2) give a short exact sequence

$$0 \to k \to C_n/B_n \to Z_{n-1} \to 0.$$

Consider the long exact sequence of cohomology groups associated to this short exact sequence:

$$\cdots \to H^{i-1}(P; Z_{n-1}) \to H^i(P;k) \to H^i(P; C_n/B_n) \to H^i(P; Z_{n-1}) \to H^{i+1}(P,k) \to \cdots.$$

The cohomology groups with coefficients in Z_{n-1} and C_n/B_n can be computed using the exact sequences in (1) and (2). Applying Exercise 28 to the exact sequence in (1), we obtain that $H^i(G; C_n/B_n) = 0$ for $i \geq 1$. Similarly aapplying Exercise 28 to the exact sequence

$$0 \to Z_{n-1} \to C_{n-1} \to \cdots \to C_0 \to k \to 0,$$

we obtain that $H^i(P;k) \cong H^{i+n}(P; Z_{n-1})$ for $i \geq 1$. Putting these calculations into the long exact sequence above, we conclude that $H^i(P;k) \cong H^{i+n+1}(P;k)$ for $i \geq 1$. This contradicts with the earlier calculation that

$$H^*(C_p \times C_p; k) \cong \begin{cases} k[x_1, x_2] & \text{if } p = 2 \\ \wedge_k(y_1, y_2) \otimes k[x_1, x_2] & \text{if } p > 2. \end{cases}$$

Hence G does not include a subgroup P isomorphic to $C_p \times C_p$. □

An alternative approach to the above proof would be using some well-known results about kG-modules. If G is a finite group and k is a field with characteristic p, then the group ring kG is self-injective (i.e. kG is an injective kG-module). This follows from the isomorphism $kG^* \cong kG$. This implies, in particular, that a finitely-generated kG-module is projective if and only if it is injective (see [8, Theorem 2.2.3]). In fact, this is true even for infinitely generated modules (see [9, Theorem 11.2]). This gives that the exact sequence in (1) splits and C_n/B_n is a projective kP-module. Splicing the second exact sequence with itself infinitely many times, we obtain a periodic projective resolution of k as a kP-module. Existence of a periodic resolution implies that for every $i \geq 1$, there is an isomorphism $H^i(P;k) \cong H^{i+n+1}(P;k)$. Again this gives a contraction.

Exercise 29. Prove that if G is a finite group and k has characteristic p, then $kG^* \cong kG$. Using this, conclude that if M is a finitely-generated kG-module, then M is projective if and only if M is injective.

Exercise 30. Show that if p is odd, and X is a finite-dimensional free C_p-CW-complex X such that $H_*(X; \mathbb{F}_p) \cong H_*(S^n; \mathbb{F}_p)$, then n is an odd number.

3 Borel Construction for Free Actions on Products of Spheres

3.1 Borel construction

Let G be a finite group. A **universal G-space**, denoted by EG, is a free G-CW-complex which is contractible (non-equivariantly). The universal G-space EG is defined uniquely up to G-homotopy equivalence. The orbit space $BG = EG/G$ is the **classifying space of** G. Since EG is contractible, the augmented chain complex $C_*(EG;k) \to k$ defines a free kG-resolution of k. This gives an isomorphism $H^*(G;k) \cong H^*(BG;k)$ which can be taken as the topological definition of group cohomology.

If X is a G-CW-complex, then G acts freely on the product space $EG\times X$ via the diagonal action $g(e,x) = (ge, gx)$. The orbit space of this action $X_G := EG \times_G X = (EG \times X)/G$ is called the **Borel construction** for the G-space X. The projection map $EG \times X \to EG$ induces a map $\pi : X_G \to BG$ on the orbit spaces. Since the quotient map $EG \to BG$ defines a principle G-bundle, $\pi : X_G \to BG$ defines a fibre bundle with fiber X (see [5, Proposition II.2.3]). Since all the spaces involved are CW-complexes. $\pi : X_G \to BG$ is a fibration with homotopy fibre X (see [21, Prop 4.48]). We call this fibration the **Borel fibration** for the G-action on X. The following observation is key for using Borel construction for free actions (see [10, Proposition 1]).

Lemma 31. *Let X be a free G-CW-complex. Then the Borel construction X_G is homotopy equivalent to the orbit space X/G.*

Proof. Consider the fibration $EG \to X_G \to X/G$ induced by the projection map $EG \times X \to X$. Since EG is contractible this gives that the map $\overline{\pi}_2 : X_G \to X/G$ is a weak equivalence. Since both of the spaces are CW-complexes, $\overline{\pi}_2$ is a homotopy equivalence. □

It is often useful to compare Borel fibrations between different group actions. Let X be a G-CW-complex. For each subgroup $H \leq G$, there is a map $i : BH \xrightarrow{Bi} BG$ and $X_H \xrightarrow{X_i} X_G$ induced by inclusion map $i : H \hookrightarrow G$. This gives a map of fibrations between the corresponding Borel fibrations:

$$\begin{tikzcd} X \arrow[r] & EG \times_G X \arrow[r] & BG \\ X \arrow[r] \arrow[u, equal] & EH \times_H X \arrow[r] \arrow[u, "X_i"'] & BH. \arrow[u, "Bi"'] \end{tikzcd}$$

Since a map of fibrations induces a morphism of spectral sequences between corresponding Serre spectral sequences, for each $r \geq 0$, there is a morphism $i^* : E_r(G) \to E_r(H)$ between the Serre spectral sequences for X_G and X_H such that i^* commutes with the differential d_r. On the E_2-page this map is of the form

$$i^* : H^*(G; H^*(X;k)) \to H^*(H; H^*(X;k))$$

and it coincides with the restriction map Res^G_H in group cohomology.

3.2 Serre spectral sequence for the Borel fibration

The Serre spectral sequence associated to the fibration $\pi : X_G \to BG$ has E_2-page

$$E_2^{s,t} = H^s(BG; H^t(X;k))$$

and it converges to $H^*(X_G;k)$. Here $H^s(BG; H^t(X;k))$ denotes the cohomology with local coefficients. The $\pi_1(BG) \cong G$ action on $H^*(X;k)$ coincides with the G-action on $H^*(X;k)$ induced by the G-action on X.

Lemma 32. *If X is a finite-dimensional free G-CW-complex, then in the Serre spectral sequence for the Borel fibration $X_G \to BG$, we have $E_\infty^{s,t} = 0$ for every s, t with $s+t > \dim X$.*

Proof. Since the G-action on X is free, by Lemma 31 the Borel construction X_G is homotopy equivalent to the orbit space X/G. Since X is a finite-dimensional G-CW-complex, the orbit space X/G is also finite-dimensional, hence $H^i(X_G;k)=0$ for $i>\dim X$. This gives that in the Serre spectral sequence $E_\infty^{i,j}=0$ for every s,t with $s+t>\dim X$. $\square$

This observation is main idea for proving restrictions on free actions on topological spaces. To illustrate this method, we now give a different proof for Smith's theorem.

Proof of Smith's theorem using the Serre spectral sequence. Let G be a finite group and X be a finite-dimensional free G-CW-complex with mod-p cohomology of S^n. Then $H^t(X;k)\cong k$ for $t=0,n$ and $H^t(X;k)=0$ when $t\neq 0,n$ where k is a field with characteristic p. As before assume contrary that G has a subgroup $P\cong C_p\times C_p$. Consider the Borel fibration for the P-action on X. Since k has characteristic p, the P action on k is trivial. This gives that $E_2^{s,t}\cong H^s(G;k)$ for $t=0,n$, and zero otherwise. Since all the differentials $d_j : E_i^{s,t}\to E_i^{s+j,t-j+1}$ are equal to zero for $2\le j\le n$, we have $E_{n+1}^{s,t}\cong E_2^{s,t}$. Consider the differential $d_{n+1}^{s,n} : E_{n+1}^{s,n}\to E_{n+1}^{s+n+1,0}$. Observe that all the higher differentials are zero, so $E_\infty\cong E_{n+2}$. Since the action is free, by Lemma 32, we have $E_\infty^{s,t}=0$ for $s+t>\dim X$. Hence the differential $d_{n+1}^{s,n}$ must be an isomorphism for $s+n>\dim X$. This gives an isomorphism $H^s(P;k)\cong H^{s+n+1}(P;k)$ for all $s>(\dim X)-n$. This is again in a contradiction with the calculation of $H^*(C_p\times C_p;k)$. $\square$

Exercise 33. (a) Show that if G is a finite group which acts freely and cellularly on a finite-dimensional CW-complex X such that $H_*(X;\mathbb{F}_p)\cong H_*(S^n\times S^n;\mathbb{F}_p)$, then there is a long exact sequence

$$\cdots\to H^i(G;k)\to H^{i+n+1}(G;M)\to H^{i+2n+2}(G,k)\to H^{i+1}(G;k)\to H^{i+n+2}(G,M)\to\cdots$$

where $M=H^n(X;k)$.
(b) Using the exact sequence in part (a) and using the calculation given in Proposition 18, prove Conner's theorem: if G acts freely on a finite-dimensional CW-complex X which has mod-p homology of $S^n\times S^n$, then G does not include $\mathbb{Z}/p\times\mathbb{Z}/p\times\mathbb{Z}/p$ as a subgroup.

Exercise 34. Let $G\cong(\mathbb{Z}/2)^r$ be an elementary abelian 2-group of rank r, and let X be a finite free G-CW-complex such that $H^*(X;\mathbb{F}_2)\cong S^n\times S^m$ with $1\le n<m$. Analyse the differentials on the Serre spectral sequence for an Borel fibration (see [28, Sect 7] for a similar analysis) and conclude that $r\le 2$.

The $p=2$ case of Theorem 4 is proved by Carlsson using the Serre spectral sequence for the corresponding Borel fibration. In the proof one of the crucial ingredients is the product structure on the Serre spectral sequence.

Proposition 35 ([24, Theorem 5.2])**.** *Let X be a G-CW-complex and k be a field with characteristic p. Then the Serre spectral sequence has a product structure $E_r^{p,q}\otimes E_r^{s,t}\to E_r^{p+s,q+t}$ such that the derivations defined by the composition*

$$H^p(G;H^q(X;k))\otimes H^s(G;H^t(X;k))\to H^{p+s}(G;H^q(X;k)\otimes H^t(X;k))$$
$$\to H^{p+s}(G;H^{q+t}(X;k))$$

where both maps are defined by the cup product. If the induced G-action on $H^*(X;k)$ is trivial, then there is an isomorphism

$$E_2^{p,q} = H^p(G; H^q(X;k)) \cong H^p(BG;k) \otimes H^q(X;k).$$

In this case the product structure on $E_2^{*,*} \cong H^p(G;k) \otimes H^q(X;k)$ is defined by $(x\otimes u)(y\otimes v) = xy \otimes uv$ where xy and uv are the cup products in $H^*(G;k)$ and $H^*(X;k)$.

3.3 Carlsson's theorem for homologically trivial actions: the $p = 2$ case.

In this section we sketch the proof of the $p = 2$ case of Carlsson's theorem stated as Theorem 4 in the introduction.

Suppose that $G \cong (\mathbb{Z}/2)^r$ and X is a finite free G-CW-complex homotopy equivalent to $(S^n)^k$ for some $n \geq 1$. Consider the Serre spectral sequence

$$H^*(G; H^*(X; \mathbb{F}_2)) \Rightarrow H^*(X_G; \mathbb{F}_2)$$

for the G-action on X. Since the action is free $X_G \cong X/G$ has finite-dimensional cohomology, so we must have $E_\infty^{p,q} = 0$ for $p+q > \dim X$. It is assumed that G acts trivially on $H^*(X;\mathbb{F}_2)$, so we have an isomorphism

$$E_2^{p,q} \cong H^p(G;\mathbb{F}_2) \otimes H^q(X;\mathbb{F}_2).$$

The spectral sequence has a product structure induced by the cup products on the cohomology groups $H^*(G;\mathbb{F}_2)$ and $H^*(X;\mathbb{F}_2)$. In this case we have $H^*(G;\mathbb{F}_2) \cong \mathbb{F}_2[x_1,\dots,x_r]$, where $|x_i| = 1$ for all i, and $H^*(X;\mathbb{F}_2) \cong \wedge_{\mathbb{F}_2}(t_1,\dots,t_k)$, where $|t_i| = n$ for all i. We will identify $E_2^{0,*}$ with $H^*(X;\mathbb{F}_2)$ and $E_2^{*,0}$ with $H^*(G;\mathbb{F}_2)$. Using the product structure we will write the elements in $E_2^{p,q}$ as a product of elements in $H^p(G;\mathbb{F}_2)$ and $H^q(X;\mathbb{F}_2)$.

Because of dimension reasons, the first nonzero differentials in this spectral sequence appear on the E_{n+1}-page, so we have $E_{n+1}^{*,*} \cong E_2^{*,*}$ and the first nonzero differentials are

$$d_{n+1}^{p,jn} : E_{n+1}^{p,jn} \to E_{n+1}^{p+n+1,jn-n}$$

for $j = 1,\dots,k$. For $i = 1,\dots,k$, let $\mu_i := d_{n+1}^{0,n}(t_i) \in H^{n+1}(G;\mathbb{F}_2)$. We call the cohomology classes $\mu_1,\dots,\mu_k$ the **k-invariants** of the G-action on X. Note that since $H^*(G;\mathbb{F}_2) \cong \mathbb{F}_2[x_1,\dots,x_r]$ we can consider the classes μ_i as polynomials of degree $n+1$ with variables $x_1,\dots,x_n$.

Lemma 36. *Let $\mu_1,\dots,\mu_k$ be the k-invariants of a free $G = (\mathbb{Z}/2)^r$-action on $X \simeq (S^n)^k$ in $\mathbb{F}_2$-coefficients. Then the cohomology classes $\mu_1,\dots,\mu_k$ have no common zeros when they are considered as polynomials in $\mathbb{F}_2[x_1,\dots,x_r]$.*

Proof. Using the isomorphism $H^1(G;\mathbb{F}_2) \cong \mathrm{Hom}_{ab}(G;\mathbb{F}_2)$, we can choose a set of generators $\{g_1,\dots,g_r\}$ for $G \cong (\mathbb{Z}/2)^r$ such that $g^i(x_j) = 1$ if $i = j$, and 0 otherwise. Note that for every element $g = g_1^{\lambda_1}\dots g_r^{\lambda_r}$ in G, we have $\mathrm{Res}^G_{\langle g\rangle} x_i = \lambda_i x$ where x is the generator of $H^1(\langle g\rangle;\mathbb{F}_2)$ dual to g. Hence for every $f(x_1,\dots,x_r)$ in $H^p(G;\mathbb{F}_2)$, and for every $g = g_1^{\lambda_1}\dots g_r^{\lambda_r}$, we have

$$\mathrm{Res}^G_{\langle g\rangle} f(x_1,\dots,x_r) = f(\lambda_1,\dots,\lambda_r).$$

If the polynomials $\mu_1, \ldots, m_k$ have a common zero, then there is an element $(\lambda_1, \ldots, \lambda_r) \in \mathbb{F}_2^r$ such that $\mu_i(\lambda_1, \ldots, \lambda_r) = 0$ for all $i = 1, \ldots, k$. If we take $g \in G$ as the element $g = g_1^{\lambda_1} \ldots g_r^{\lambda_r}$, then $\mathrm{Res}^G_{\langle g \rangle} \mu_i = 0$ for all $i = 1, \ldots, k$. We now argue that this gives a contradiction.

For every subgroup $H \leq G$, this is a morphism of spectral sequences between the Serre spectral sequences for G-space X and its restriction to H. This gives a morphism between corresponding Serre spectral sequences. In particular, there is a homomorphism $i^* : E^{*,*}_{n+1}(G) \to E^{*,*}_{n+1}(H)$ of spectral sequences induced by the inclusion map $i : H \to G$. Since i^* induces the identity map on $E^{0,*}_{n+1} \cong H^*(X; \mathbb{F}_2)$ and it is the restriction map on $E^{*,0}_{n+1} \cong H^*(G; \mathbb{F}_2)$, we obtain that the differential $d^{0,n}_{n+1}(H)$ for the Borel fibration $X_H \to BH$ takes t_i to $\mathrm{Res}^G_H \mu_i$ for each i. If we apply this to the subgroup $\langle g \rangle$ such that $\mathrm{Res}^G_{\langle g \rangle} \mu_i = 0$ for all $i = 1, \ldots, k$, we obtain that in the Serre spectral sequence for the $\langle g \rangle$-action on X, we have $d^{0,n}_{n+1} = 0$. Using the product structure and the Leibniz rule, this gives that $d^{p,q}_{n+1} = 0$ for all p, q. The higher dimensional differentials will also vanish by a similar argument. Thus in the Serre spectral sequence for the $\langle g \rangle$-action on X, we have

$$E^{*,*}_\infty \cong E^{*,*}_{n+1} \cong E^{*,*}_2 \cong H^*(X; \mathbb{F}_2) \otimes H^*(\langle g \rangle; \mathbb{F}_2)$$

which implies that $\bigoplus_{p+q=N} E^{p,q}_\infty \neq 0$ for all N. This contradicts with the conclusion of Lemma 32. □

The fact that the k-invariants $\mu_1, \ldots, \mu_k$ have no common zeros by itself is not enough to conclude that $r \leq k$. For example $\mu_1 = x_1^2 + x_1 x_2 + x_2^2$ has no zeros as a polynomial in $\mathbb{F}_2[x_1, x_2]$, but in this case $r = 2 > k = 1$. So we need more restrictions on the k-invariants $\mu_1, \ldots, \mu_k$.

Lemma 37. *Let $\mu_1, \ldots, \mu_k$ be the k-invariants of a free $G = (\mathbb{Z}/2)^r$-action on $X \simeq (S^n)^k$ in $\mathbb{F}_2$-coefficients. Then the ideal $I = (\mu_1, \ldots, \mu_k) \subseteq \mathbb{F}_2[x_1, \ldots, x_r]$ generated by the k-invariants of the action is closed under the Steenrod operations, i.e, for each i, j, we have $\mathrm{Sq}^j(\mu_i) \in I$.*

Proof. An element $x \in E^{0,q}_2 = H^0(BG, H^q(X; \mathbb{F}_2))$ is called **transgressive** if $d_j(x) = 0$ for $j = 2, \ldots, q$. In this case we say x transgresses to the element $y = d_{q+1}(x) \in E^{q+1,0}_{q+1}$. There is a theorem which says that if $x \in H^q(X, \mathbb{F}_2)$ is G-invariant and transgressive, then $\mathrm{Sq}^j(x)$ is also G-invariant and transgressive, and we have $d_{q+j+1}(\mathrm{Sq}^j(x)) = \mathrm{Sq}^j(y)$ (see [21, p. 540]). This theorem is sometimes called the Kudo's transgression theorem. Applying this theorem to the Serre spectral sequence for the $G = (\mathbb{Z}/2)^r$ action on $X = (S^n)^k$, we obtain that for each $i \in \{1, \ldots, k\}$, and $0 \leq j \leq n$, $\mathrm{Sq}^j(\mu_i) = 0$ in $E^{n+j+1,0}_{n+j+1}$. The only nonzero differential hitting $E^{n+j+1,0}_{n+1}$ is $d^{j,n}_{n+1}$, so there are elements $\gamma_1, \ldots, \gamma_k \in H^j(G; \mathbb{F}_2)$ such that

$$\mathrm{Sq}^j(\mu_i) = d^{j,n}_{n+1}\Big(\sum_l \gamma_l t_l\Big) = \sum_l \gamma_l \mu_l.$$

Hence, for each i, we have $\mathrm{Sq}^j(\mu_i) \in (\mu_1, \ldots, \mu_k)$. □

Now, the proof of Theorem 4 follows from the following Theorem due to Serre [29].

Theorem 38. *Let $I \subseteq \mathbb{F}_2[x_1, \ldots, x_r]$ be an ideal invariant under Steenrod operations. Then the variety $V(I)$ defined by the ideal I over $k = \overline{\mathbb{F}}_2$, the algebraic closure of $\mathbb{F}_2$, is union of linear subspaces rational over $\mathbb{F}_2$.*

Serre's theorem implies that if I is a homogeneous ideal generated by $f_1, \dots, f_k$ in $\mathbb{F}_2[x_1, \dots, x_r]$, then $f_1, \dots, f_k$ have no common zeros over $\mathbb{F}_2$ if and only if they have no common zeros over $k = \overline{\mathbb{F}}_2$. This implies, in particular, that $k \geq r$. Applying this to the k-invariants $\mu_1, \dots, \mu_k$ in $H^*(G; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \dots, x_r]$, we obtain the inequality $r \leq k$. This completes the proof of Theorem 4 for $p = 2$. $\square$

Remark 39. Note that the above proof does not hold when $p > 2$. In this case the Steenrod operations send the generators $t_1, \dots, t_k$ of $H^n(X; \mathbb{F}_p)$ to the dimensions higher than $2n$, so one can not directly conclude that the ideal generated by the k-invariants $\mu_1, \dots, \mu_k$ is a Steenrod closed ideal. Carlsson proves Theorem 4 for $p > 2$ in [11] by constructing a chain map

$$C_*(X) \to \otimes_{i=1}^k C_*(\mu_i)$$

from the chain complex of a finite G-CW-complex $X \simeq (S^n)^k$ with k-invariants $\mu_1, \dots, \mu_k$ to a tensor product of chain complexes $C_*(\mu_i)$ which are algebraic homology spheres with k-invariants $\mu_1, \dots, \mu_k$.

Later a similar argument was used by Benson and Carlson [3] to give a different proof of Theorem 4. Benson and Carlson uses L_ζ-modules and the theory of support varieties for kG-modules. Because of time constraints I will not be able to cover these proofs in my lectures. We refer the reader to original papers cited above for these proofs.

4 Method of Exponents and Browder's Theorem

4.1 Tate Hypercohomology

Let G be a finite group. A **complete resolution** for G is an acyclic chain complex

$$(F_*, \varepsilon) : \cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} F_{-1} \xrightarrow{\partial_{-1}} F_{-2} \to \cdots$$

of free $\mathbb{Z}G$-modules together with a homomorphism $\varepsilon : F_0 \to \mathbb{Z}$ such that

$$F_* : \cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Note that if F_* is a complete resolution, then $\ker \partial_{-1} = \operatorname{im} \partial_0$ is isomorphic to $\mathbb{Z}$ via the map ε. So the boundary homomorphism ∂_0 splits as $\partial_0 = \eta\varepsilon$, where $\eta : \mathbb{Z} \to F_{-1}$ is the injection mapping $\mathbb{Z}$ into the kernel of ∂_{-1}. The map η together with boundary maps in negative dimensions defines a coresolution

$$0 \to \mathbb{Z} \xrightarrow{\eta} F_{-1} \xrightarrow{\partial_{-1}} F_{-2} \xrightarrow{\partial_{-2}} F_{-3} \to \cdots$$

Since each F_i is a free $\mathbb{Z}G$-module, this is a coresolution of $\mathbb{Z}$ with relatively injective $\mathbb{Z}G$-modules (see [7, Sect VI.2]).

Definition 40. The **Tate cohomology** of a finite group G with coefficients in a $\mathbb{Z}G$-module M is defined by

$$\widehat{H}^n(G; M) := H^n(\operatorname{Hom}_{\mathbb{Z}G}(F_*, M))$$

for all $n \in \mathbb{Z}$, where F_* is a complete resolution for G.

One can show that a complete resolution for G is unique up to an augmentation preserving chain homotopy. So the definition of Tate cohomology is independent of the chosen resolution for G. From the definition it follows easily that for any $\mathbb{Z}G$-module M,

$$\widehat{H}^n(G;M) \cong \begin{cases} H^n(G;M) \text{ if } n > 0 \\ H_n(G;M) \text{ if } n < -1 \end{cases}$$

and there is an exact sequence

$$0 \to \widehat{H}^{-1}(G;M) \to H_0(G;M) \xrightarrow{\overline{N}_G} H^0(G;M) \to \widehat{H}^0(G;M) \to 0.$$

where $\overline{N}_G : M_G \to M^G$ is the norm map defined by $\overline{N}_G([m]) = \sum_{g\in G} gm$. In particular we have the following:

Lemma 41. *For every finite group G, we have $\widehat{H}^0(G;\mathbb{Z}) \cong \mathbb{Z}/|G|\mathbb{Z}$.*

Using a complete diagonal approximation, one can define a product structure on Tate cohomology which coincides with the cup product for group cohomology in positive dimensions (see [7, Sect VI.5]).

Exercise 42. Show that if F is a free $\mathbb{Z}G$-module, then $\widehat{H}^i(G;F) = 0$ for all $i \in \mathbb{Z}$.

We now define the Tate hypercohomology of a finite group G with coefficients in a chain complex C_* of $\mathbb{Z}G$-modules. For this we need to extend the definition of the Hom-functor that we defined earlier. Suppose C_* and D_* are chain complexes over $\mathbb{Z}G$ with differentials ∂^C and ∂^D, respectively. Let $\mathscr{H}om_G(C_*, D_*)$ denote the cochain complex with n-cochains

$$\mathscr{H}om_G(C_*, D_*)^n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}G}(C_i, D_{i-n})$$

whose coboundary maps defined by $\delta^n(f) = \partial^D f - (-1)^n f \partial^C$. If D_* is a chain complex concentrated at dimension 0 with $D_0 = M$, then $\mathscr{H}om_G(-, D_*)$ is naturally equivalent to the functor $\mathrm{Hom}_G(-, M)$ that we defined earlier.

Definition 43. Let G be a finite group and C_* be a chain complex of $\mathbb{Z}G$-modules. The **Tate hypercohomology** of G with coefficients in C_* is defined by

$$\widehat{H}^i(G;C_*) := H^i(\mathscr{H}om_G(F_*, C_*))$$

for all $i \in \mathbb{Z}$, where F_* is a complete $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

For a chain complex C_*, the (left) k-shifted complex $\Sigma^k C_*$ is defined to be the chain complex where $(\Sigma^k C_*)_i = C_{i-k}$ with boundary map $(\Sigma^k \partial)_i = (-1)^k \partial_{i-k}$. For a cochain complex C^* we define the (left) k-shifted cochain complex by $(\Sigma^k C^*)^i = C^{i+k}$ with coboundary maps $(\Sigma^k \delta)^i = (-1)^k \delta^{i+k}$. It is easy to see that

$$\mathscr{H}om_G(C_*, \Sigma^k D_*) \cong \Sigma^k \mathscr{H}om_G(C_*, D_*).$$

From this we obtain that $\widehat{H}^i(G;\Sigma^k C_*) \cong \widehat{H}^{i+k}(G;C_*)$. Therefore, if C_* is a chain complex concentrated at dimension n, then $\widehat{H}^i(G;C_*) \cong \widehat{H}^{i+n}(G;C_n)$.

Lemma 44. *Given a short exact sequence of chain complexes*

$$0 \to C_* \to D_* \to E_* \to 0$$

of $\mathbb{Z}G$-modules, there is a long exact sequence of the following form

$$\cdots \to \widehat{H}^i(G, C_*) \to \widehat{H}^i(G, D_*) \to \widehat{H}^i(G, E_*) \to \widehat{H}^{i+1}(G, C_*) \to \cdots .$$

The following observation is crucial for proofs using Tate hypercohomology.

Proposition 45. *If F_* is a bounded chain complex of free $\mathbb{Z}G$-modules, then $\widehat{H}^i(G; F_*) = 0$ for all $i \in \mathbb{Z}$.*

Proof. By shifting the complex if necessary we can assume that F_* is a chain complex of the form

$$0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to 0.$$

There is a short exact sequence of chain complexes

$$0 \to F'_* \to F_* \to F''_* \to 0$$

where F'_* is the chain complex

$$0 \to F_{d-1} \to \cdots \to F_1 \to F_0 \to 0.$$

and F''_* is the chain complex concentrated at dimension d with $F''_d = F_d$. By induction $\widehat{H}^i(G; F'_*) = 0$ for all $i \in \mathbb{Z}$, and by Exercise 42, $\widehat{H}^i(G; F''_*) = 0$ for all $i \in \mathbb{Z}$. Hence now the result follows from Lemma 44. □

4.2 Browder's Theorem

The **exponent** of a finite abelian group A is defined as the smallest positive integer n such that $na = 0$ for all $a \in A$. We denote the exponent of A by $expA$. A nonnegative chain complex C_* of $\mathbb{Z}G$-modules is said to be **connected** if $H_0(C_*) \cong \mathbb{Z}$. A nonnegative chain complex C_* is **finite-dimensional** if there is an integer d such that $C_d \neq 0$ and $C_i = 0$ for all $i > d$. In this case we say d is the **dimension of** C_*. By convention $C_* = 0$ is also finite-dimensional with dimension -1.

The main purpose of this section is to prove the following theorem due to Browder [6].

Theorem 46 (Browder [6])**.** *Let C_* be a nonnegative chain complex of free $\mathbb{Z}G$-modules. If C_* is connected and finite-dimensional, then $|G|$ divides $\prod_{j=1}^{\dim C_*} expH^{j+1}(G; H_j(C_*))$.*

Proof. We will give a proof of Browder's theorem using a hypercohomology spectral sequence that converges to the Tate hypercohomology $\widehat{H}^*(G; C_*)$. To construct this spectral sequence, consider the double complex $D^{p,q} = \mathrm{Hom}_{\mathbb{Z}G}(F_p, C_{-q})$ where the vertical and horizontal differentials are given by $\delta_v = \mathrm{Hom}(-, \partial^C_*)$ and $\delta_h = \mathrm{Hom}(\partial^F_*, -)$. For this construction we assume that C_* is a finite-dimensional, nonnegative chain complex of $\mathbb{Z}G$-modules. Then the total complex $\mathrm{Tot}D^{*,*}$ is the chain complex with

$$\mathrm{Tot}^n D^{*,*} = \bigoplus_{p+q=n} D^{p,q} = \bigoplus_{p+q=n} \mathrm{Hom}_{\mathbb{Z}G}(F_p, C_{-q})$$

and $\delta^n = \delta_v - (-1)^n \delta_h$. Note that the total complex $\mathrm{Tot} D^{*,*}$ is isomorphic to the cochain complex $\mathscr{H}om_G(F_*, C_*)$ defined earlier. Filtering this double complex with respect to the index p and then with respect to the index q, we obtain two spectral sequences

$$^I E_2^{p,q} = \widehat{H}^p(G; H_{-q}(C_*)) \Rightarrow \widehat{H}^{p+q}(G; C_*)$$

$$^{II} E_1^{p,q} = \widehat{H}^q(G; C_{-p}) \Rightarrow \widehat{H}^{p+q}(G; C_*).$$

If C_* is a chain complex of free $\mathbb{Z}G$-modules, then $^{II} E_1^{p,q} = \widehat{H}^q(G; C_{-p}) = 0$ for all p, q. This implies that $\widehat{H}^n(G; C_*) = 0$ for all n. This gives a strong restriction on the first spectral sequence. Note that since C_* is nonnegative, $^I E_r^{*,*}$ is nonzero only on the lower half plane. If C_* is connected, then $^I E_2^{p,0} = \widehat{H}^p(G; \mathbb{Z})$. In particular, $^I E_2^{0,0} \cong \mathbb{Z}/|G|\mathbb{Z}$ by Lemma 41. For each $r \geq 2$, the differentials out of $^I E_r^{0,0}$ are of the form

$$d_r^{0,0} : {}^I E_r^{0,0} \to {}^I E_r^{r,-r+1}.$$

Since the generator of $^I E_2^{0,0} \cong \mathbb{Z}/|G|\mathbb{Z}$ does not survive to E_∞-page (because the E_∞-page is zero), we can conclude that $|G|$ divides the product $\prod_{r=2}^{\dim C_*+1} exp E_r^{r,-r+1}$. Since for each $r \geq 2$, $E_r^{r,-r+1}$ is a subquotient of $E_2^{r,-r+1} = \widehat{H}^r(G; H_{r-1}(C_*))$, we know that $exp E_r^{r,-r+1}$ divides $exp \widehat{H}^r(G; H_{r-1}(C_*))$. Using this, we obtain that $|G|$ divides the product

$$\prod_{r=2}^{\dim C_*+1} exp \widehat{H}^r(G; H_{r-1}(C_*)) = \prod_{j=1}^{\dim C_*} exp H^{j+1}(G; H_j(C_*)).$$

□

As a corollary of Theorem 46, Browder gives a proof for Theorem 4. Note that if $C_* = C_*(X)$ for some finite-dimensional G-CW-complex X homotopy equivalent to $(S^n)^k$, then $H_*(X)$ is nonzero in exactly k many positive dimensions. If $G \cong (\mathbb{Z}/p)^r$ and M is a trivial $\mathbb{Z}G$-module, then $exp H^i(G; M)$ divides p for all $i \geq 1$. So, from the conclusion of Theorem 46, one obtains that $|G| = p^r$ divides p^k, which gives $r \leq k$.

The method of exponents are also used to study G-CW-complexes with non trivial isotropy subgroups and for these type of actions a theorem similar to Theorem 46 is proved by Adem in [1].

In [2], Adem and Browder proves Theorem 5. The main algebraic input for the proof is certain dimension inequalities for the representations of $G = (\mathbb{Z}/p)^r$ over p-local integers $\mathbb{Z}_{(p)}$. The main topological ingredient is the following theorem.

Theorem 47 ([2, Thm 1.1])**.** *Let $G = (\mathbb{Z}/p)^r$ act freely on an orientable $\mathbb{Z}_{(p)}$-homology manifold X with $H^*(X; \mathbb{Z}_{(p)}) \cong H^*((S^n)^k; \mathbb{Z}_{(p)})$, then*

$$\dim H_n(X; \mathbb{F}_p)^G \geq \mathrm{rk} H$$

where $H \leq G$ is the subgroup of elements in G acting trivially on $H_*(X; \mathbb{Z}_{(p)})$.

The proof of this theorem also uses an argument with exponents of cohomology groups. We refer the reader to the original paper [2] for this proof.

5 Free Actions on Products of Spheres at High Dimensions

5.1 Habegger's Theorem

Let C_* and D_* be two chain complexes of $\mathbb{Z}G$-modules. We say that D_* is an extension of C_* by a finite length chain complex of free modules if there is short exact sequence of chain complexes either of the form $0 \to C_* \to D_* \to F_* \to 0$, or of the form $0 \to F_* \to D_* \to C_* \to 0$, where F_* is a finite length chain complex of free modules. The chain complexes C_* and D_* are **freely equivalent** if there is a sequence of chain complexes $C_* = E^0_*, E^1_*, \dots, E^n_* = D_*$ such that either E^i_* is an extension of E^{i-1}_*, or E^{i-1}_* is an extension of E^i_* by a finite length chain complex of free modules. As a corollary of Proposition 45 and Lemma 44, we have:

Corollary 48. *If two chain complexes C_* and D_* are freely equivalent, then $\widehat{H}^i(G, C_*) \cong \widehat{H}^i(G, D_*)$ for all i.*

In [17, p. 433-434], Habegger uses a technique to "glue" homology groups of a chain complex at different dimensions. This technique will be crucial in the proof of Theorem 8. Before we state Habegger's theorem, we recall the definition of syzygies of modules.

Definition 49. For every positive integer n, the n-th syzygy of a $\mathbb{Z}G$-module M is defined as the kernel of ∂_{n-1} in a partial resolution of the form

$$P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \to M \to 0$$

where $P_0, \dots, P_{n-1}$ are projective $\mathbb{Z}G$-modules. We denote the n-th syzygy of M by $\Omega^n M$ and by convention we take $\Omega^0 M = M$.

The n-th syzygy of a module M is well-defined only up to stable equivalence. Recall that two $\mathbb{Z}G$-modules M and N are called **stably equivalent** if there are projective $\mathbb{Z}G$-modules P and Q such that $M \oplus P \cong N \oplus Q$. Well-definedness of syzygies up to stable equivalence follows from a generalization of Schanuel's lemma (see [7, p. 193]). If M and N are two stably equivalent $\mathbb{Z}G$-modules, then $\widehat{H}^i(G, M) \cong \widehat{H}^i(G, N)$ for all i. Since we are interested in cohomology groups with coefficients in these modules, we will ignore the fact that syzygies are well-defined only up to stable equivalence and treat $\Omega^n M$ as a unique module depending only on M and n.

Theorem 50 (Habegger [17]). *Let C_* be a chain complex of $\mathbb{Z}G$-modules and n, m are integers such that $m < n$. If $H_k(C_*) = 0$ for all k with $m < k < n$, then C_* is freely equivalent to a chain complex D_* such that*

(i) $H_i(D_*) = H_i(C_*)$ for every $i \neq n, m$;

(ii) $H_m(D_*) = 0$, and;

(iii) there is an exact sequence of $\mathbb{Z}G$-modules

$$0 \to H_n(C_*) \to H_n(D_*) \to \Omega^{n-m} H_m(C_*) \to 0.$$

Proof. Let $F_{n-1} \to \cdots \to F_m \to H_m(C_*) \to 0$ be an exact sequence of $\mathbb{Z}G$-modules where $F_m, \dots, F_{n-1}$ are free modules. Consider the following diagram

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_{m+1} & \longrightarrow & F_m & \longrightarrow & H_m(C_*) & \longrightarrow & 0 \\
 & & & & & & & & & & & & \big\downarrow{\scriptstyle id} & & \\
\cdots & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_{m+1} & \longrightarrow & Z_m & \longrightarrow & H_m(C_*) & \longrightarrow & 0
\end{array}$$

where Z_m denotes the group of m-cycles in C_*. Since all F_i's are free and the bottom row has no homology at dimensions less than n, the identity map extends to a chain map $f'_* : F_* \to C'_*$ between these chain complexes. This gives a chain map $f_* : F_* \to C_*$ as follows

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_{m+1} & \longrightarrow & F_m & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f_{m+1}} & & \downarrow{\scriptstyle f_m} & & \downarrow & & \\
\cdots & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_{m+1} & \longrightarrow & C_m & \longrightarrow & C_{m-1} & \longrightarrow & \cdots .
\end{array}$$

where the maps $f_i : F_i \to C_i$ for $i \geq m+1$ are the same as the maps f'_* in the first diagram above, and the map $f_m : F_m \to C_m$ is defined as the composition

$$F_m \xrightarrow{f'_m} Z_m \hookrightarrow C_m.$$

Now, let D_* be the mapping cone of $f_* : F_* \to C_*$. There is a short exact sequence

$$0 \to C_* \to D_* \to \Sigma F_* \to 0,$$

so C_* is freely equivalent to D_*. Consider the corresponding long exact sequence of homology groups

$$\cdots \longrightarrow H_i(F_*) \xrightarrow{f_*} H_i(C_*) \longrightarrow H_i(D_*) \longrightarrow H_{i-1}(F_*) \longrightarrow \cdots .$$

Assume first that $n > m+1$. Then F_* has at least two terms and its homology is nonzero only at two dimensions $n-1$ and m. So, $H_i(C_*) \cong H_i(D_*)$ for all i such that $i \neq m, m+1, n-1, n$. At dimension m, the map $f_* : H_m(F_*) \to H_m(C_*)$ is an isomorphism, so we get $H_m(D_*) = H_{m+1}(D_*) = 0$. At dimension $n-1$, we have $H_{n-1}(C_*) = 0$, so we get $H_{n-1}(D_*) = 0$. We also have a short exact sequence of the form

$$0 \longrightarrow H_n(C_*) \longrightarrow H_n(D_*) \longrightarrow H_{n-1}(F_*) \longrightarrow 0.$$

Since $H_{n-1}(F_*) \cong \Omega^{n-m}(H_m(C_*))$, this gives the desired result.

If $n = m+1$, then F_* has a single term F_m, so we have a sequence of the form

$$0 \longrightarrow H_n(C_*) \longrightarrow H_n(D_*) \longrightarrow F_m \xrightarrow{f_*} H_m(C_*) \longrightarrow H_m(D_*) \longrightarrow 0.$$

Since f_* is surjective by construction, we conclude that $H_m(D_*) = 0$ and there is a short exact sequence of the form

$$0 \longrightarrow H_n(C_*) \longrightarrow H_n(D_*) \longrightarrow \Omega^1(H_m(C_*)) \longrightarrow 0$$

as desired. □

5.2 Proof of Okutan-Yalçın Theorem

In this section we give a proof of Theorem 8 stated in the introduction. Let $G = (\mathbb{Z}/p)^r$ and k, l be positive integers. Suppose that G acts freely and cellularly on some CW-complex X homotopy equivalent to $S^{n_1} \times \cdots \times S^{n_k}$ where $|n_i - n_j| \leq l$ for all i, j. Let $n = \max\{n_i : i = 1, \ldots, k\}$ and let $a_i = n - n_i$ for all i. Consider the cellular chain complex $C_*(X)$ of the CW-complex X. The complex $C_*(X)$ is a nonnegative, connected, and finite-dimensional chain complex of free $\mathbb{Z}G$-modules and has nonzero homology only at the following dimensions other than dimension zero:

$$
\begin{array}{ll}
(1) & n-a_1, n-a_2, \ldots, n-a_k \\
(2) & 2n-a_1-a_2, 2n-a_1-a_3, \ldots, 2n-a_{k-1}-a_k \\
 & \vdots \\
(j) & jn-(a_1+\cdots+a_j), \ldots, jn-(a_{k-j+1}+\cdots+a_k) \\
 & \vdots \\
(k) & kn-(a_1+a_2+\cdots+a_k).
\end{array}
$$

If $n > lk$, then we have $n > a_1+\cdots+a_k$ which implies that for all j, the dimensions listed on the j-th row are strictly larger than the dimensions listed on the previous rows. Since this fact is important for our argument, we will assume that the integer N in the statement of the theorem satisfies $N > lk$ to guarantee that this condition holds.

Now we can apply Habegger's argument given in Theorem 50 to glue step by step all the homology groups at the dimensions listed on the j-th row above to the homology at dimension jn for all $j = 1, \ldots, k$. The resulting complex D_* is a connected, finite-dimensional chain complex of free $\mathbb{Z}G$-modules which has homology only at dimensions $0, n, 2n, \ldots, kn$. Let $M_j := H_{jn}(D_*)$ for all $j = 1, \ldots, k$. Note that by construction M_j is a finitely-generated $\mathbb{Z}G$-module for all j since syzygies of finitely-generated $\mathbb{Z}G$-modules are finitely-generated when G is a finite group.

To estimate the exponents of cohomology groups with coefficients in M_j's we need the following observation due to Pakianathan [26].

Lemma 51. *Let $G = (\mathbb{Z}/p)^r$ and M be a finitely-generated $\mathbb{Z}G$-module. Then, there is an integer N such that the exponent of $H^i(G, M)$ divides p for all $i \geq N$.*

Proof. By Theorem 7.4.1 in [15, p. 87], $H^*(G, M)$ is a finitely-generated module over the ring $H^*(G, \mathbb{Z})$. Let $u_1, ..., u_k$ be homogeneous elements generating $H^*(G, M)$ as an $H^*(G, \mathbb{Z})$-module and let $N = 1 + \max_j\{\deg u_j\}$. If $i \geq N$ and $x \in H^i(G, M)$, then we can write $x = \Sigma_{j=1}^k \alpha_j u_j$ for some homogeneous elements α_j in $H^*(G, \mathbb{Z})$ with $\deg \alpha_j \geq 1$ for all j. Since $expH^i(G, \mathbb{Z})$ divides p for all $i \geq 1$, we have $p\alpha_j = 0$ for all j. Hence we obtain $px = 0$ as desired. ☐

Now we can apply Lemma 51 to find an integer N_j for each j such that if $i \geq N_j$, then $expH^i(G, M_j)$ divides p. Suppose that for a fixed $G = (\mathbb{Z}/p)^r$, k, and l, there are only finitely many possibilities for $\mathbb{Z}G$-modules M_j's up to stable equivalence. If this is the case, then the proof can be completed by the following argument:

Let $N_j^{\max}$ be the maximum of N_j's over all possible M_j. Then for each j such that if $i \geq N_j^{\max}$, then $expH^i(G, M_j)$ divides p for all possible M_j's that may occur. Let $N = \max_j N_j^{\max}$. By Theorem 46, we have $|G| = p^r$ divides

$$\prod_{j=1}^{k} H^{jn+1}(G; H_{jn}(D_*)) = \prod_{j=1}^{k} H^{jn+1}(G; M_j).$$

So, if $n \geq N$, then p^r divides p^k which gives $r \leq k$ as desired.

Hence to complete the proof, it only remains to show that for fixed $G = (\mathbb{Z}/p)^r$, k, and l, there are only finitely many possibilities for $\mathbb{Z}G$-modules M_j's up to stable equivalence. To show this, first note that for a fixed l, there are finitely many k-tuples $(a_1, ..., a_k)$ with the property that $0 \le a_i \le l$ for all i. So we can assume that we have a fixed k-tuple $(a_1, \dots, a_k)$. Let us also fix an integer j and show there are only finitely many possibilities for $M_j = H_{jn}(D_*)$.

Let $s_1 < \cdots < s_m$ be a sequence of integers such that $\{jn - s_1, \dots, jn - s_m\}$ is the set of all distinct dimensions on the j-th row of the above diagram. Note that the complex D_* is constructed with the repeated usage of Theorem 50, so the module $M_j = H_{jn}(D_*)$ has a filtration

$$0 = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = M_j$$

such that $K_i/K_{i-1} \cong \Omega^{s_i}(A_i)$ where $A_i = H_{jn-s_i}(X)$. For all i, the module A_i is a $\mathbb{Z}$-free $\mathbb{Z}G$-module with $\mathrm{rk}_{\mathbb{Z}} A_i \le \binom{k}{j}$. We can now use the following theorem due to Jordan and Zassenhaus.

Theorem 52 (Cor 79.12 in [14, p. 563])**.** *Let G be a finite group and d a positive integer. Then there are only finitely many $\mathbb{Z}$-free $\mathbb{Z}G$-modules with dimension $\le d$.*

Since $\mathrm{rk}_{\mathbb{Z}} A_i \le \binom{k}{j}$, by Jordan-Zassenhaus theorem, there are finitely many possibilities for A_i's up to isomorphism. Now we will inductively show that there are also only finitely many possibilities for K_i's up to stable equivalence. For $i = 1$, we have $K_1 = \Omega^{s_1}(A_1)$ so this follows from the fact that there are only finitely many possibilities for A_1 and that syzygies are well-defined up to stable equivalence. For $i > 1$, consider the following short exact sequence:

$$0 \longrightarrow K_{i-1} \longrightarrow K_i \longrightarrow \Omega^{s_i} A_i \longrightarrow 0.$$

By induction we know that there are only a finite number of possibilities for K_{i-1}'s up to stable equivalence. By a similar argument as above, the same is true for $\Omega^{s_i}(A_i)$. The extensions like the ones above are classified by the ext-group $\mathrm{Ext}^1_{\mathbb{Z}G}(\Omega^{s_i}(A_i), K_{i-1})$ and since both modules are $\mathbb{Z}$-free, these ext-groups are well-defined up to stable equivalence. So, it remains to show that

$$\mathrm{Ext}^1_{\mathbb{Z}G}(\Omega^{s_i}(A_i), K_{i-1}) \cong \mathrm{Ext}^{s_i+1}_{\mathbb{Z}G}(A_i, K_{i-1})$$

is a finite group. Since both A_i and K_{i-1} are finitely generated, $\mathrm{Ext}^{s_i+1}_{\mathbb{Z}G}(A_i, K_{i-1})$ is a finitely generated abelian group. Moreover, since A_i is $\mathbb{Z}$-free, it has an exponent divisible by $|G|$. So, $\mathrm{Ext}^{s_i+1}_{\mathbb{Z}G}(A_i, K_{i-1})$ is a finite group. Hence there are only a finite number of possibilities for K_i's up to stable equivalence. This completes the proof that there are only finitely many possibilities for M_j's up to stable equivalence, Hence the proof of Theorem 8 is complete.

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