APPLICATIONS OF THE UCT FOR C*-ALGEBRAS

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Contents

1. An introduction to the Universal Coefficient Theorem / Eilers

This is an introduction to the UCT of Rosenberg and Schochet, its importance in particular in classification, its various generalizations, and the open questions concerning the class of C^*-algebras to which it applies.

Classical

Rosenberg and Schochet proved that a particular class of C^*-algebras satisfy the Universal Coefficient Theorem in KK-theory. Through out the paper, they assume that the C^*-algebra A is separable nuclear, and the C^*-algebra B has a countable approximate unit.

Let $\mathcal{N}$ be the smallest full subcategory of the separable nuclear C^*-algebras which contains the separable Type I C^*-algebras and is closed under strong Morita equivalence, inductive limits, extensions, and crossed products by $\mathbb{R}$ and $\mathbb{Z}$.

Theorem (UCT)**.** *Let $A \in \mathcal{N}$. Then there is a short exact sequence*

$$0 \to Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK_*(A,B) \xrightarrow{\gamma} Hom(K_*(A), K_*(B)) \to 0$$

which is natural in each variable. The map γ has degree 0 and the map δ has degree 1.

Proof. We consider

$$\gamma(A,B) : KK_*(A,B) \to Hom(K_*(A), K_*(B))$$

- If $K_*(B)$ is injective, $I \triangleleft A$, and two out of $\gamma(I,B), \gamma(A,B), \gamma(A/I,B)$ are isomorphisms, so is the last.
- If $K_*(B)$ is injective, $A = \varinjlim A_i$, and all $\gamma(A_i, B)$ are isomorphisms, so is $\gamma(A,B)$.
- If $K_*(B)$ is injective then $\gamma(C_0(X), B)$ is an isomorphism.
- If $K_*(B)$ is injective and A is type I then $\gamma(A,B)$ is an isomorphism.
- If $K_*(B)$ is injective, and $A \in \mathcal{N}$, then $\gamma(A,B)$ is an isomorphism.
- For any σ-unital B there is $\varphi : B \to D$ with $K_*(D)$ injective and $\varphi_* : K_*(B) \to K_*(D)$ injective.

□

Remark: The authors noted that it was quite possible that the UCT holded for completely arbitrary separable C^*-algebras A, assuming that B has a countable approximate unit. An interesting open problem was to determine whether the UCT might in fact holded for all separable C^*-algebras. This was shown to be equivalent to the question: is every separable C^*-algebra KK-equivalent to a commutative C^*-algebra? However, later work of Skandalis showed that this was not the case, though this may be true for separable nuclear C^*-algebras.

Proposition (RS)**.** *If $A \in \mathcal{N}$, then A is KK-equivalent to some $C_0(X)$.*

Theorem (Skandalis). *The following are equivalent for a separable A (not necessarily nuclear!)*

(1) The UCT holds for A and any B
(2) A is KK-equivalent to some $C_0(X)$
(3) If $K_*(B) = 0$, then $KK(A, B) = 0$

and there is a non-nuclear A for which they are false.

Modern

Theorem (Elliott). *For A and B AT-algebras of real rank zero, we have*

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K} \Leftrightarrow (K_*(A), K_*(A)_+) \cong (K_*(B), K_*(B)_+)$$

Theorem (UMCT, Dadarlat- Loring). *For $A \in \mathcal{N}$ we have*

$$\begin{array}{ccccc} Pext(K_*(A), K_*(B)) & & & & Hom(K_*(A), K_*(B)) \\ & \searrow & & \nearrow & \\ & & KK(A, B) & & \\ & \nearrow & & \searrow & \\ Ext(K_*(A), K_*(B)) & & & & Hom_\Lambda(\underline{K}(A), \underline{K}(B)) \end{array}$$

Theorem (Dadarlat-Loring). *For A and B AD-algebras of real rank zero, we have*

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K} \Leftrightarrow (\underline{K}(A), \underline{K}(A)_+) \cong (\underline{K}(B), \underline{K}(B)_+)$$

Theorem (Kirchberg-Phillips). *Suppose A and B are simple, separable, nuclear, purely infinite C^*-algebras. If A and B are KK-equivalent, then $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.*

Theorem (Kirchberg-Phillips). *Suppose A and B are simple, separable, nuclear, purely infinite C^*-algebras with $A, B \in \mathcal{N}$. Then*

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K} \Leftrightarrow K_*(A) \cong K_*(B).$$

Contemporary

Theorem (Tikuisis-White-Winter). *If A is separable, nuclear and satisfies the UCT, then any amenable trace on A is quasidiagonal.*

Theorem (Dadarlat). *If A is separable, exact, residually finite-dimensional and satisfies the UCT, then A is AF-embeddable.*

New classes

A satisfies the UCT when

- $A = C^*(G)$ for certain amenable groupoids G (Tu)
- A may be locally approximated with UCT subalgebras (Dadarlat)

- $A = C^*_\pi(G)$ for nilpotent group (Eckharadt-Gillaspy)
- A has a Cartan subalgebra (Barlak-Li).

Localizations The UCT holds for all nuclear C^*-algebras if $\mathcal{O}_2$ is unique with $K_*(\mathcal{O}_2) = 0$ among the purely infinite, nuclear C^*-algebras (Kirchberg).

Theorem (Kirchberg)**.** *Suppose A and B ar separable, nuclear, purely infinite C^*-algebras with*

$$Prim(A) \cong X \cong Prim(B).$$

If A and B are $KK(X)$-equivalent, then $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.

Theorem (Meyer-Nest,Bentmann-Köhler)**.** *Suppose A and B are separable, nuclear, purely infinite C^*-algebras with $A, B \in \mathcal{N}$ and*

$$Prim(A) \cong X \cong Prim(B)$$

with X a finite accordion space. Then,

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K} \Leftrightarrow K_*(X, A) \cong K_*(X, B).$$

2. Elements of KK theory and UCT / Dadarlat

In this section we assume that A is always a separable C^*-algebra and B a σ-unital C^*-algebra.

Object of Interest: To study topological invariants of C^*-algebras. A first attempt would be to consider the $*$-homomorphisms $\phi : A \to B$ but in many cases they don't seem to be enough to implement topological information about A and B. KK-theory is a theory of generalised $*$-homomorphisms that gives valuable topological information. It is a bivariant theory in the sense that it combines $K-$theory and K-homology. Let's see the "trivial" but enlighting case of $KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$.

(1) **K-theory picture:** Consider the Murray-von Neumann classes of finite dimensional projections $e, f \in \mathcal{L}(H)$. Then $dim(e) - dim(f) = [e] - [f] \in KK(\mathbb{C}, \mathbb{C})$.
(2) **K-homology picture:** Consider the homotopy classes of Fredholm operators $T \in \mathcal{L}(H)$. Then $\text{Index}(T) = dim \ker(T) - dim \ker(T^*) = dim(e) - dim(f) \in KK(\mathbb{C}, \mathbb{C})$ for some projections e, f as above that $\ker(T) = Im(e)$ and $\ker(T^*) = Im(f)$.

Hilbert C^*-modules

Consider E to be a right B-module.

Definition. *E is a right Hilbert B-module if there exists a map $< \cdot, \cdot >: E \times E \to B$ such that:*

- *it is linear in the second variable and anti-linear in the first;*
- $< \eta, \xi b > = < \eta, \xi > b$;
- $< \eta, \xi >^* = < \xi, \eta >$;
- $< \eta, \eta > \geq 0$ *and equals* 0 *iff* $\eta = 0$;
- *E is complete w.r.t* $\|\xi\| = \| < \xi, \xi > \|^{\frac{1}{2}}$.

Examples:

- $B = \mathbb{C}$, then E is a Hilbert space.
- $E = B$ with $< b_1, b_2 > = b_1^* b_2$.
- $E = B^n$ with $< a, b > = \sum_{i=1}^n a_i^* b_i$.
- $E = eB^n \subset B^n$ for $e \in M_n(B)$ with the above inner product. If e is a projection then eB^n is a projective Hilbert B-module since $eB^n \oplus (1_n - e)B^n = B^n$. Consider X to be a compact Hausdorff space and $B = C(X)$. Then any complex vector bundle over X corresponds to a projective $C(X)$-module.
- $E \triangleleft B$ with the above inner product.

Theorem (Absorption Theorem / Kasparov)**.** *If E is a countably generated Hilbert $B-$module then* $E \oplus H_B \cong H_B$ *where* $H_B = \{b \in \prod_{i=1}^{\infty} B : \sum_{i=1}^{\infty} b_i^* b_i < \infty\} = \bigoplus_{i=1}^{\infty} B$.

Let E, F be two Hilbert $B-$modules. Define

$$\mathcal{L}(E,F) = \{T : E \to F : \text{ linear, adjointable}\}$$

and $\mathcal{K}(E,F) = \{\theta_{\xi,\eta} = \eta < \xi, \cdot >: \ \xi \in E, \eta \in F\}$ where they are considered to be the "compact operators". If $E = F$ then these are $C^* - algebras$.

Facts:

- $\mathcal{K}(B) \cong B$, by multiplication operators $M_x : B \to B, b \mapsto xb$ for $x \in B$.
- $\mathcal{L}(B) \cong M(B)$.
- **Kasparov:** $M(\mathcal{K}(E)) \cong \mathcal{L}(E)$. In particular, $M(B \otimes K) \cong \mathcal{L}(H_B)$.

Definition (According to Atkinson's Theorem)**.** *An* $S \in \mathcal{L}(E^{(0)}, E^{(1)})$ *is "Fredholm" if there exists* $T \in \mathcal{L}(E^{(1)}, E^{(0)})$ *such that* $1 - TS \in \mathcal{K}(E^{(0)})$ *and* $1 - ST \in \mathcal{K}(E^{(1)})$.

If $Im(S)$ is a closed B-module, then $\ker S$ and $\ker S^*$ are f.g projective B-modules and hence we can define $Index(S) = [\ker S] - [\ker S^*] \in K_0(B)$.

Remark. *Actually we don't need to assume that* $Im(S)$ *is closed since in general there is a compact perturbation* S' *of* $S \oplus 1 : E^{(0)} \oplus H_B \to E^{(1)} \oplus H_B$ *such that* $Im(S)$ *is closed. Hence* $Index(S) = Index(S') \in K_0(B)$.

KK-groups

Definition (Kasparov bimodules)**.** *Let* A, B *be graded. An* (A, B)*-bimodule is a countably generated graded Hilbert* C^**-module over* B *acted upon by* A *through a grading preserving* $*$*-homomorphism* $\pi : A \to \mathcal{L}(E)$*. A Kasparov* (A, B)*-bimodule is a triple* (E, π, F) *where* E *is an* $(A, B)-$*bimodule,* $F \in \mathcal{L}(E)$ *is of odd degree such that for every* $a \in A$

- $[\pi(a), F] \in \mathcal{K}_{\mathcal{B}}(E)$;
- $\pi(a)(F^2 - 1) \in \mathcal{K}_{\mathcal{B}}(E)$;
- $\pi(a)(F^* - F) \in \mathcal{K}_{\mathcal{B}}(E)$.

The triples are called degenerate if the bullets are identically zero. We denote the set of Kasparov $(A, B)-$*bimodules by* $\mathcal{E}(A, B)$.

We say that two Kasparov (A, B)-bimodules are isomorphic ($\simeq$) if there is an isomorphism of Hilbert B- modules that intertwines the gradings, the representations and the operators of the triples. Denote by $B[0, 1] = C([0, 1], B)$ and define a homotopy between two elements $x_0, x_1 \in \mathcal{E}(A, B)$ as an element $y \in \mathcal{E}(A, B[0, 1])$ such that $(ev_0)_*(y) \simeq x_0$ and $(ev_1)_*(y) \simeq x_1$. We consider the transitive closure of homotopy and get an equivalence relation on $\mathcal{E}(A, B)$ which we denote by $\sim$. Degenerate Kasparov bimodules are equivalent to zero. It is weaker than isomorphism. There is also another notion of homotopy on $\mathcal{E}(A, B)$ called operator homotopy. In this we focus entirely on the operators of the triples and we say that x_0, x_1 are operator homotopic if there is a norm-continuous path $\mathcal{F}_t = (G, \rho, F_t)$ in $\mathcal{E}(A, B)$ such that $\mathcal{F}_0 \simeq x_0$ and $\mathcal{F}_1 \simeq x_1$. With an additional assumption we can turn it into an equivalence relation which we denote by $\approx$. This is stronger than $\sim$ and weaker than isomorphism. We can form the direct sum of two Kasparov $(A, B)-$bimodules:

$$(E, \pi, F) \oplus (E', \pi', F') = (E \oplus E', \pi \oplus \pi', F \oplus F')$$

which is well-defined on $\mathcal{E}(A, B)/\sim$. Moreover, we can form the inverse of $x = [E, \pi, F]$ as $x' = [E^{opp}, \pi^{opp}, -F]$ since $x \oplus x' = [\text{degenerate}] = [0]$ where the operator homotopy is given by $\mathcal{G}_t = (E \oplus E^{opp}, \pi \oplus \pi^{opp}, G_t)$ and

$$G_t = \begin{pmatrix} F cos\frac{\pi}{2}t & sin\frac{\pi}{2}t \\ sin\frac{\pi}{2}t & -F cos\frac{\pi}{2}t \end{pmatrix}$$

Actually one can prove that the operator homotopy and homotopy induce isomorphic abelian groups and we define the

$$\begin{aligned} KK(A, B) &= \mathcal{E}(A, B)/\sim \\ &\cong \mathcal{E}(A, B)/\approx . \end{aligned}$$

We note that $KK(\mathbb{C}, A) = K_0(A)$ and $KK(A, \mathbb{C}) = K^0(A)$.

Kasparov Product

Kasparov product is a bilinear operation $KK(D, A) \times KK(A, B) \to KK(D, B)$ which has the following properties:

- naturality;
- associativity;
- functoriality in all possible ways.

Roughly speaking, this corresponds to a "product" of Fredholm operators $F \# F'$ such that $Index(F \# F') = Index(F) Index(F')$. This is the case for $KK(\mathbb{C}, \mathbb{C})$.

With that we can talk about the KK-category (an additive category) and the KK-functor from $C^* \to KK$ is:

- homotopy invariant in both variables;
- stable;
- split-exact.

Actually a theorem of Higson characterises the KK-functor as being universal amongst the homotopy invariant, stable and split exact additive functors on the category of separable C^*-algebras, in the sense that if $F : C^* \to A$ (A is an additive category) is such a functor then there exists a unique functor $F' : KK \to A$ such that $F' \circ KK = F$. Any such functor satisfies Bott periodicity.

Given a semi-split short exact sequence of separable C^*-algebras

$$0 \to J \to A \to A/J \to 0$$

we get a six-term exact sequence in both variables for any separable C^*-algebra D in the other variable. We can extend this to $\sigma-$unital $J, A, A/J$ if we consider D on the left and to $\sigma-$unital D if we consider it to the right.

The KK-functor is σ-additive in the first variable;

$$KK(\bigoplus_{n=1}^{\infty} A_n, B) \cong \prod_{n=1}^{\infty} KK(A_n, B)$$

but not in the second variable in general.

The KK-functor is contravariant in the first variable and covariant in the second, as does the Hom functor. Actually, the Kasparov product allows us to relate the two functors:

Note: From now on, A and B are trivially graded.

Consider the homomorphism

$$\otimes(\cdot) : KK_*(A, B) \to Hom(KK_*(D, A), KK_*(D, B))$$

and for $D = \mathbb{C}$ we get the map

$$\gamma(A, B) : KK_*(A, B) \to Hom(K_*(A), K_*(B)).$$

Let $\mathcal{N}$ be the smallest full subcategory of the separable nuclear C^*-algebras which contains the separable Type I C^*-algebras and is closed under strong Morita equivalence, inductive limits, extensions, and crossed products by $\mathbb{R}$ and $\mathbb{Z}$. The Universal Coefficient Theorem (UCT) is the following:

Theorem (Rosenberg, Schochet)**.** *Let A and B be separable C^*-algebras, with $A \in \mathcal{N}$. Then there is a short exact sequence*

$$0 \to Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK_*(A, B) \xrightarrow{\gamma} Hom(K_*(A), K_*(B)) \to 0$$

The map γ has degree 0 and δ has degree 1. The sequence is natural in each variable, and splits unnaturally. So if $K_*(A)$ is free or $K_*(B)$ is divisible, then γ is an isomorphism.

Write $KK_1(A, B) = KK(A, B \otimes \mathbb{C}_1)$ and

$$\gamma^1(A, B) : KK_1(A, B) \to Hom(K_0(A), K_1(B)) \oplus Hom(K_1(A), K_0(B)).$$

We use the fact that $KK_1(A, B) \cong Ext(A, B)$. For an extension

$$x : 0 \to B \to E \to A \to 0$$

we have $\gamma^1(A, B)(x) = \delta_0(x) \oplus \delta_1(x)$ where $\delta_0(x) : K_0(A) \to K_1(B)$ and $\delta_1(x) : K_1(A) \to K_0(B)$ are the boundary maps of the corresponding six-term exact sequence in K-theory. Hence if $x \in \ker \gamma^1(A, B)$ then $\delta_0(x)$ and $\delta_1(x)$ are zero and consequently we get

$$0 \to K_*(B) \to K_*(E) \to K_*(A) \to 0,$$

that is, an element in $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$. Denote this map by

$$k^1 : \ker \gamma^1(A, B) \to Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)).$$

This is a degree 0 map and one of the goals, for such A and B, is to prove that k^1 is bijective. Then by suspending again we get

$$k^0 : \ker \gamma^0(A,B) \to Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$$

which is a degree 1 map and hence $k = k^0 \oplus k^1$ is a degree 1 map. Thus we get the map δ.

Definition. *A separable C^*-algebra A is in the UCT class if for every σ-unital C^*-algebra B we have that $\gamma(A,B)$ is surjective and k is bijective.*

The proof of UCT will be in two parts: a special case where $K_*(B)$ is divisible and a general case by using geometric resolution of C^*-algebras.

Special Case of UCT

First a quick recap on Milnor lim^1-sequence and divisible groups:

Let

$$\ldots \to G_n \xrightarrow{f_{n-1}} G_{n-1} \to \ldots \to G_2 \xrightarrow{f} G_1$$

be a projective system of abelian groups. Then we get

$$0 \to \varprojlim G_n \to \prod_{n=1}^{\infty} G_n \xrightarrow{I-S} \prod_{n=1}^{\infty} G_n \to \varprojlim{}^1 G_n \to 0$$

where $I - S : (\lambda_n)_{n=1}^{\infty} \mapsto (\lambda_n - f_n(\lambda_{n+1}))_{n=1}^{\infty}$ and $\varprojlim{}^1 G_n := coker(I-S)$.

With this, for an inductive limit $A_1 \to A_2 \to \ldots$ of nuclear C^*-algebras we get the mapping telescope sequence:

$$0 \to \varprojlim{}^1 KK_1(A_n, B) \to KK(\varinjlim A_n, B) \to \varprojlim KK(A_n, B) \to 0.$$

Definition. *An abelian group G is divisible if $G = nG$ for every $n \geq 1$.*

Proposition. *The following are equivalent:*

- *G is divisible;*
- *G is injective[1];*
- *$Hom(\cdot, G)$ is exact;*
- *$Ext^1_{\mathbb{Z}}(\cdot, G) = 0$.*

Suppose that

$$0 \to B \to X \to Y \to 0$$

is a short exact sequence of abelian groups. Then for any abelian group A we get

$$\begin{array}{ccccccc} 0 \longrightarrow & Hom(A,B) & \longrightarrow & Hom(A,X) & \longrightarrow & Hom(A,Y) & \\ & Ext^1_{\mathbb{Z}}(A,B) & \longrightarrow & Ext^1_{\mathbb{Z}}(A,X) & \longrightarrow & Ext^1_{\mathbb{Z}}(A,Y) & \longrightarrow 0 \end{array}$$

[1]Injectivity means than we can extend homomorphisms that map into G from subgroups to groups in a unique way.

by using the snake lemma. If X, Y are injective; this means, we get an injective resolution for B, then $Ext^1_{\mathbb{Z}}(A, X)$ and $Ext^1_{\mathbb{Z}}(A, Y)$ vanish. And this completes the recap.

Suppose that $K_*(B)$ is divisible, then $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) = 0$. Hence we investigate if $\gamma(A, B)$ is an isomorphism. To do that we compare the two functors $KK_*(\cdot, B)$ and $Hom(K_*(\cdot), K_*(B))$. Note that both are σ-additive cohomology theories on separable C^*-algebras and general C^*-algebras respectively.

Then let $\mathcal{N}'$ be the smallest class of separable C^*-algebras A for which $\gamma(A, B)$ is an isomorphism. Then $\mathcal{N}'$ contains $\mathbb{C}$ since $KK^i(\mathbb{C}, B) = K_i(B) = Hom(\mathbb{Z}, K_i(B))$. Using the five lemma one can show that if two of A, J and A/J are in $\mathcal{N}'$ then so is the third and also an application of the five lemma to the lim^1-sequences yields that $\mathcal{N}'$ is closed under inductive limits. Finally, the naturality of the intersection product shows that $\mathcal{N}'$ is closed under KK-equivalence. Hence $\mathcal{N}'$ contains $\mathcal{N}$ and the special case of the UCT is proved.

General case of UCT

Proposition. *Let A be a separable C^*-algebra such that $\gamma(A, B)$ is an isomorphism for all B separable with $K_*(B)$ divisible. Then A is in the UCT class.*

Proof. We will make use of geometric injective resolutions. Given B separable C^*-algebra we construct a semisplit extension

$$0 \longrightarrow D \xrightarrow{\alpha} C \xrightarrow{\beta} SB \longrightarrow 0$$

such that $K_*(D)$ and $K_*(C)$ are divisible and $\beta_* = 0$. Hence the six-term exact sequence unsplices to two short exact sequences

$$0 \longrightarrow K_{i+1}(SB) \longrightarrow K_i(D) \xrightarrow{\alpha_*} K_i(C) \longrightarrow 0$$

Applying the $Hom(K_*(A), \cdot)$ functor we get

$$\begin{array}{l} 0 \longrightarrow Hom(K_*(A), K_*(B)) \longrightarrow Hom(K_*(A), K_*(D)) \\ \qquad\qquad \swarrow Hom(1,\alpha_*) \\ Hom(K_*(A), K_*(C)) \longrightarrow Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow 0 \end{array}$$

$$\begin{aligned} ker Hom(1, \alpha_*) &= Hom(K_*(A), K_*(B)) \\ coker Hom(1, \alpha_*) &= Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)). \end{aligned}$$

Now we consider the six-term exact sequence with the following squares:

$$
\begin{array}{ccccc}
Hom(K_*(A),K_*(D)) & \xrightarrow{Hom(1,\alpha_*)} & Hom(K_*(A),K_*(C)) & & \\
{\scriptstyle \gamma(A,D)}\uparrow & & {\scriptstyle\gamma(A,C)}\uparrow & & \\
KK_*(A,D) & \xrightarrow{\omega} & KK_*(A,C) & \longrightarrow & KK_*(A,SB) \\
\uparrow & & & & \downarrow \\
KK_{*+1}(A,SB) & \longleftarrow & KK_{*+1}(A,C) & \xleftarrow[\omega]{} & KK_{*+1}(A,D) \\
 & & \downarrow{\scriptstyle\gamma'(A,C)} & & \downarrow{\scriptstyle\gamma'(A,D)} \\
 & & Hom(K_*(A),K_*(C)) & \xrightarrow{Hom(1,\alpha_*)} & Hom(K_*(A),K_*(D))
\end{array}
$$

where γ' is the odd degree version of γ.

The six-term exact sequence unsplices and we get

$$0 \longrightarrow coker\omega \longrightarrow KK_{*+1}(A,SB) \longrightarrow ker\omega \longrightarrow 0$$

and since the corner squares commute and $\gamma(A,C), \gamma(A,D)$ are isomorphisms we get that

$$ker\omega \cong kerHom(1,\alpha_*)$$
$$coker\omega \cong cokerHom(1,\alpha_*).$$

One can see that the map $KK_{*+1}(A,SB) \to Ker\omega \cong kerHom(1,\alpha_*) = Hom(K_*(A),K_*(B))$ is $\gamma(A,B)$ and thus we get

$$0 \to Ext^1_{\mathbb{Z}}(K_*(A),K_*(B)) \xrightarrow{\delta} KK_*(A,B) \xrightarrow{\gamma(A,B)} Hom(K_*(A),K_*(B)) \to 0$$

where δ is the inverse of k.

$\square$

So we have reduced the problem to the construction of a geometric injective resolution for any separable B.

Construction of Geometric resolution

Given an abelian group G we can construct an injective resolution easily. Let $f : F_0 \to G$ be a homomorphism of a free abelian group onto G and let $F_1 = kerf$. Then F_1 is also free and $G \cong F_0/F_1$. Let g be the composition homomorphism

$$G \longrightarrow F_0/F_1 \xrightarrow{\cong} (F_0 \otimes \mathbb{Q})/F_1 := I_0$$

Then g is injective and I_0 is divisible as a quotient of the divisible $F_0 \otimes \mathbb{Q}$. Let $I_1 = I_0/G$. Then

$$0 \longrightarrow G \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0$$

is the required injective resolution. The proof for the geometric resolution uses the same idea but the construction starts on the C^*-algebraic level. Recall that given a $*$-homomorphism $f : A \to B$ the mapping cone

$$Cf = \{(\xi, a) \in BI \oplus A : \xi(0) = 0, \xi(1) = f(a)\}.$$

There is a natural map $Cf \to A$ given by $(\xi, a) \mapsto a$ and the resulting "mapping cone sequence"

$$0 \longrightarrow SB \longrightarrow Cf \longrightarrow A \longrightarrow 0$$

is semisplit where the c.p map $A \to Cf$ is given by $a \mapsto (a, (1-t)f(a))$.

Theorem. *Let B be a separable C^*-algebra. Then there exists a separable C^*-algebra D whose K-groups are divisible and a $*$-homomorphism $f : SB \to D$ such that the induced map*

$$K_{*+1}(B) = K_*(SB) \xrightarrow{f_*} K_*(D)$$

is injective.

The theorem implies the existence of a geometric resolution by just considering the mapping cone sequence for f:

$$0 \longrightarrow SD \longrightarrow Cf \longrightarrow SB \longrightarrow 0$$

and since f_* is injective, the six-term exact sequence in K-theory degenerates to

$$0 \longrightarrow K_*(SB) \longrightarrow K_*(D) \longrightarrow K_{*+1}(Cf) \longrightarrow 0$$

and since $K_{*+1}(Cf)$ is a quotient of a divisible group then itself is divisible.

Proof. It suffices to assume that B is unital since the unitalisation map $B \to B^+$ induces an inclusion in the K-theory. So let B be a unital C^*-algebra. Then, just like in the proof of the Künneth theorem we can construct a projective resolution for B. That is, we construct a commutative C^*-algebra F with $K_*(F)$ being free abelian and a $*$-homomorphism $r : F \to B \otimes K$ such that it induces a surjective map on K-theory. The mapping cone sequence

$$0 \longrightarrow SB \otimes K \longrightarrow Cr \xrightarrow{s} F \longrightarrow 0$$

yields a degenerate K-theory sequence of the form

$$0 \longrightarrow K_*(Cr) \xrightarrow{s_*} K_*(F) \xrightarrow{r_*} K_*(B \otimes K) \longrightarrow 0$$

since r_* is surjective and r_* is actually the composition

$$K_*(F) \xrightarrow{\partial} K_{*+1}(SB \otimes K) \xrightarrow{\cong} K_*(B \otimes K).$$

Let N be a unital AF-algebra with $K_0(N) = \mathbb{Q}$. (Take for instance $N = \varinjlim (M_2 \otimes \dots M_{n!})$ with the obvious maps.) Let $t : F \to F \otimes N$ given by $t(x) = x \otimes 1$. Then the induced map $t_* : K_*(F) \to K_*(F \otimes N) \cong K_*(F) \otimes \mathbb{Q}$ is injective and $K_*(F \otimes N)$ is divisible. To see the last isomorphism one can use the Künneth theorem. The mapping cone sequence for $ts : Cr \to F \otimes N$ has associated degenerate K-theory sequence

$$0 \longrightarrow K_*(Cr) \xrightarrow{(ts)_*} K_*(F \otimes N) \longrightarrow K_*(C_{ts}) \longrightarrow 0$$

for $(ts)_* = t_* s_*$ is injective, and we are half way there since we get an injective resolution of $K_*(Cr)$. Using the naturality of the cone construction we get a map of mapping cone sequences

$$\begin{array}{ccccccccc} SCr & \longrightarrow & SF & \longrightarrow & Cs & \longrightarrow & Cr & \xrightarrow{s} & F \\ \downarrow & & \downarrow & & \downarrow u & & \downarrow & & \downarrow t \\ SCr & \longrightarrow & SF \otimes N & \longrightarrow & Cts & \longrightarrow & Cr & \xrightarrow{s} & F \otimes N \end{array}$$

a hence a commuting diagram of short exact sequences

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & K_*(Cr) & \longrightarrow & K_*(F) & \longrightarrow & K_*(Cs) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow t_* & & \downarrow u_* & & \\ 0 & \longrightarrow & K_*(Cr) & \longrightarrow & K_*(F \otimes N) & \longrightarrow & K_*(Cts) & \longrightarrow & 0 \end{array}$$

The five lemma implies that $u_* : K_*(Cs) \to K_*(Cts)$ is injective. Actually, we can related $SB \otimes K$ to Cs in the following way: Since $SB \otimes K$ is the kernel of $s : Cr \to F$ we get a short exact sequence

$$0 \longrightarrow SB \otimes K \xrightarrow{v} Cs \longrightarrow CF \longrightarrow 0$$

where the map $Cs \to CF$ is given simply by $(\xi, a) \mapsto \xi$ and its kernel identifies with $SB \otimes K$. Therefore, we get an isomorphism $v_* : K_*(SB \otimes K) \to K_*(Cs)$ and if we consider an inclusion $w : SB \to SB \otimes K$ we define f to be the composition

$$SB \xrightarrow{w} SB \otimes K \xrightarrow{v} Cs \xrightarrow{u} C_{ts} := D.$$

Then $f_* = u_* v_* w_*$ is injective and $K_*(D)$ is divisible. This completes the proof.

□

Universal Multi-Coefficient Theorem

We now describe the UMCT of Dadarlat and Loring and give a sketchy proof of the fact that a separable C^*-algebra A satisfies the UCT if and only if it satisfies the UMCT. A quick recap: A finitely generated abelian group is of the form $\mathbb{Z}^k \oplus \mathbb{Z}_{p_1} \oplus \ldots \mathbb{Z}_{p_r}$ where p_i are prime numbers. Let $P = \{\text{powers of prime numbers}\} \cup \{0\}$. We define K-theory with coefficients; that is , we define groups $K_0(A; \mathbb{Z}_p)$ and $K_1(A; \mathbb{Z}_p)$.

Take $p \in P \setminus \{0\}$ and consider the diagonal inclusion $i : \mathbb{C} \to M_p(\mathbb{C})$. Let I_p be the mapping cone for i and define $I_p^0 = I_p$ and $I_p^1 = SI_p$. For the mapping cone we get a short exact sequence

$$0 \longrightarrow SM_p(\mathbb{C}) \longrightarrow I_p \xrightarrow{i} \mathbb{C} \longrightarrow 0$$

to which we apply $KK(\cdot, A)$ and we get the six-term exact sequence

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A;\mathbb{Z}_p) \\ \uparrow & & & & \downarrow \\ K_1(A;\mathbb{Z}_p) & \longleftarrow & K_1(A) & \longleftarrow & K_1(A) \end{array}$$

Goal: To study the map $KK(A,B) \to Hom(K_*(A;\mathbb{Z}_p), K_*(B;\mathbb{Z}_p))$ that comes from the Kasparov product.

From now on $p, q \in P$ and $i \in \{0,1\}$.

Define $\underline{K}(A) := \bigoplus_{p,i} KK_*(I_p^i, A)$ and let $\Lambda = \bigoplus_{p,q} KK_*(I_p, I_q)$. Λ is a ring with structure from $KK_*(\bigoplus_p I_p, \bigoplus_q I_q) \supset \Lambda$. Then $\underline{K}(A)$ is a left Λ-module with multiplication coming from the Kasparov product. Finally, we define the map

$$\Gamma : KK(A,B) \to Hom_\Lambda(\underline{K}(A), \underline{K}(B)).$$

This will be the map that corresponds to γ in the UCT.

Theorem (UMCT)**.** *Let A, B be C^*-algebras. Suppose that $A \in \mathcal{N}$ and B is σ-unital. Then there is a short exact sequence*

$$0 \to Pext(K_*(A), K_*(B)) \to KK(A,B) \xrightarrow{\Gamma} Hom(\underline{K}(A), \underline{K}(B)) \to 0$$

which is natural in every variable.

Note: $Pext(K_*(A), K_*(B))$ is the subgroup of $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$ consisting of pure extensions; that is, extensions

$$0 \to K_*(B) \to G \xrightarrow{p} K_*(A) \to 0$$

such that for every finitely generated subgroup H of $K_*(A)$ there is a homomorphism $j : H \to G$ such that $p \circ j = id$. Realising $KK(A,B)$ as extensions of SA by B one can prove, as in the UCT case, that $\ker \Gamma = Pext(K_*(A), K_*(B))$.

Lemma. *If A belongs to the UCT class and $K_*(A)$ is finitely generated then Γ is an isomorphism of groups for every σ-unital B.*

Proof. From the assumptions we may assume that $A \sim_{KK} \bigoplus_{p,i} I_p^i$ where p varies in a finite set. By additivity we may further assume that $A = I_p^i$. Now,

$$\Gamma : KK(I_p^i, B) \to Hom_\Lambda(\underline{K}(I_p^i), \underline{K}(B)).$$

Let $x_p^i = 1 \in KK(I_p^i, I_p^i)$. An element $a \in Hom_\Lambda(\underline{K}(I_p^i), \underline{K}(B))$ is completely determined by the image of $x_p^i \mapsto a(x_p^i) \in KK(I_p^i, B)$. Since the Kasparov product respects the unit we get that $\Gamma(a(x_p^i)) = a$.

□

Conversely:

Proposition. *If a separable A is in the UMCT class then A is in the UCT class.*

Proof. For that, we use the fact that A satisfies the UCT if and only if $KK(A, B) = 0$ for every separable B with $K_*(B) = 0$. Using the the mapping cone sequence above and the six-term exact sequence for $KK(\cdot, B)$ we get that $K_*(B; \mathbb{Z}_p) = 0$ and hence $\underline{K}(B) = 0$. Using the UMCT short exact sequence we get that $KK(A, B) = 0$. □

Finally, we introduce a new topological invariant that is defined on pairs of C^*-algebras (separable/σ-unital) that is homotopy invariant in each variable, stable and split-exact, and that is a also a polish group. First a quick recap:

We consider extensions in terms of Busby invariants. So, let $\tau : A \to \mathcal{Q}(B)$ be such. We say that it is trivial if it lifts to a $*$-homomorphism to $M(B)$. We say that τ is stably trivial if there is a trivial τ' such that $\tau \oplus \tau'$ is trivial. We say that τ is approximately trivial if there is a sequence $(\tau)_n$ of trivial Busby invariants such that $\|\tau_n(a) - \tau(a)\| \to 0$. We say that τ is stably approximately trivial if there is a trivial τ' such that $\tau \oplus \tau'$ is approximately trivial. Finally, two extensions τ_1 and τ_2 are stably unitary equivalent if there is a unitary $u \in \mathcal{Q}(B)$ and a trivial extension τ' such that $u^*(\tau_1 \oplus \tau')u = \tau_2 \oplus \tau'$. We denote by $Ext(A, B)$ the set of stable unitary equivalence classes of extensions of A by $B \otimes K$. Let $Ext^{-1}(A, B)$ be the invertible elements.

Denote by $\mathcal{T}(A, B)$ the set equivalence classes of stably approximately trivial extensions. There is a metric on $Ext(A, B)$ where $\mathcal{T}(A, B)$ corresponds to the closure of the stably trivial extensions; $\mathcal{T}(A, B) = \overline{0}$. One can prove that the Kasparov product is continuous with respect to this topology and hence see that $Ext^{-1}(A, B) \cap \mathcal{T}(A, B)$ is a group.

Definition. *Let A be separable and B be σ-unital. Then we define*

$$KL(A, B) := Ext^{-1}(SA, B)/Ext^{-1}(SA, B) \cap \mathcal{T}(SA, B).$$

If A is in the UCT class then $Pext(K_*(A), K_*(B) \cong Ext^{-1}(SA, B) \cap \mathcal{T}(SA, B)$ and by identifying $KK(A, B)$ with $Ext^{-1}(SA, B)$ we get that $KL(A, B) \cong Hom_\Lambda(\underline{K}(A), \underline{K}(B))$.

3. A new proof of the Kirchberg-Philips theorem/ Gabe

Theorem. *(Kirchberg-Phillips) Let A, B be two separable, nuclear, purely infinite, simple C^*-algebras. Then:*

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K} \text{ iff } A \sim_{KK} B.$$

Observation: If A, B satisfy the UCT then $A \sim_{KK} B$ is equivalent to $K_*(A) \cong K_*(B)$ by an application of the five lemma.

Corollary. *If A, B satisfy the conditions of the Kirchberg-Phillips theorem and in addition they are in the UCT class then*

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K} \text{ iff } K_*(A) \cong K_*(B).$$

Goal: Prove the Kirchberg-Phillips theorem more elementary by transfering the "black box" of understanding to the $\mathcal{O}_2$-embedding theorem.
Method: Existence + Uniqueness + $\mathcal{O}_2$-embedding theorem.

Existence Part: Lift a KK-element to a $*$-homomorphism.

Theorem. *Let A be a separable, nuclear C^*-algebra, B be a σ-unital C^*-algebra with a properly infinite full projection and $\alpha \in KK(A, B)$. Then there exists a full[1] $*$-homomorphim $\varphi : A \to B$ such that $KK(\varphi) = \alpha$.*

We use the Cuntz picture of KK-theory; that is, we tranfer all the information of a Fredholm module to the representation and

$$KK(A, B) = \{(\varphi, \psi) : A \to M(B \otimes \mathbb{K}), \varphi(a) - \psi(a) \in B \otimes \mathbb{K} \ \forall a \in A\}/\sim.$$

Fact: If $\theta : A \to B$ is a full $*$-homomorphism, we construct an infinite repeat of θ:

$$\begin{aligned} \theta_\infty &= \theta \oplus \theta \ldots \oplus \theta \oplus \ldots \\ &= \theta \otimes 1_{\mathbb{K}} \end{aligned}$$

and $\theta_\infty : A \to M(B \otimes \mathbb{K})$. Then any $\alpha \in KK(A, B)$ is of the form $[\varphi, \theta_\infty]$.

Lemma (1). *There exists a continuous unitary path $(u_t)_{t\in[0,\infty)}$ in $(\mathcal{O}_2 \otimes \mathbb{K})^\sim$ such that*

- $u_0 = 1$;
- *$u_t^*(1 \otimes e_{11})u_t$ is an approximate identity for $\mathcal{O}_2 \otimes \mathbb{K}$;*
- *$u_t x$ converges as $t \to \infty$, for every $x \in \mathcal{O}_2 \otimes \mathbb{K}$.*

[1] It means that $\overline{B\varphi(a)B} = B$ for every $a \in A \setminus \{0\}$. It implies simplicity of $*$-homomorphisms.

Proof. The idea is that there exists some $v \in \mathcal{U}_0(\mathcal{O}_2 \otimes M_3)$ such that

$$v^* \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} v = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$

□

Lemma (2)**.** *Suppose that $\mathcal{O}_2 \hookrightarrow M(B \otimes \mathbb{K})$ is a unital embedding. Then there is an induced unital embedding $(\mathcal{O}_2 \otimes \mathbb{K})^\sim \hookrightarrow M(B \otimes \mathbb{K})$ and we have an analogous version of Lemma 1: There exists a continuous unitary path $(u_t)_{t \in [0,\infty)}$ in $(\mathcal{O}_2 \otimes \mathbb{K})^\sim \hookrightarrow M(B \otimes \mathbb{K})$ such that*

- $u_0 = 1$;
- $u_t^*(1 \otimes e_{11})u_t$ *is an approximate identity for $B \otimes \mathbb{K}$;*
- $u_t x u_t^*$ *converges as $t \to \infty$, for every $x \in B \otimes \mathbb{K}$.*

Lemma (3)**.** *Suppose that $\theta : A \to B$ is a $*$-homomorphism such that there exists a unital embedding $\mathcal{O}_2 \hookrightarrow M(B) \cap \theta(A)'$. Then the unitary path $(u_t)_{t \in [0,\infty)}$ in $(\mathcal{O}_2 \otimes \mathbb{K})^\sim \hookrightarrow M(B \otimes \mathbb{K}) \cap \theta_\infty(A)'$ from Lemma 2 has the property:*
For every Cuntz pair (Φ, θ_∞), there exists a $*$-homomorphism $\phi : A \to B$ such that

$$u_t \Phi(a) u_t^* \to \phi(a) \oplus \theta(a) \oplus \theta(a) \oplus \dots \quad \forall a \in A.$$

Proof. Write

$$u_t \varphi(a) u_t^* = u_t(\Phi(a) - \theta_\infty(a))u_t^* + u_t \theta_\infty(a) u_t^*$$

and the first summand converges since $\Phi(a) - \theta_\infty(a) \in B \otimes \mathbb{K}$ and the second is $\theta_\infty(a)$ because $\theta_\infty(a)$ commutes with u_t. Denote the limit by $\psi(a)$. For every $b \in B \otimes \mathbb{K}$ we have

$$\begin{aligned} \|(1 - 1 \otimes e_{11})u_t b u_t^*\| &= \|u_t^*(1 - 1 \otimes e_{11})u_t b\| \\ &= \|b - u_t^*(1 \otimes e_{11})u_t b\| \to 0. \end{aligned}$$

As a result $(1 - 1 \otimes e_{11})\psi(a) = 0 \oplus \theta(a) \oplus \theta(a) \oplus \dots = \psi(a)(1 - 1 \otimes e_{11})$. Therefore we let $\phi(a) = \psi(a)(1 \otimes e_{11})$.

□

Proof of existence: Since B has a properly infinite full projection, a standard trick implies that $D = \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \mathbb{K} \subset B$, and $\overline{DBD} = B$. For our θ we consider the composition

$$A \xhookrightarrow{\iota \otimes 1 \otimes e_{11}} D \hookrightarrow B$$

where ι is a $\mathcal{O}_2$-embedding. For the embedding of $\mathcal{O}_2$ in $M(B)$ we consider the following composition:

$$\mathcal{O}_2 \xhookrightarrow{1 \otimes id \otimes 1} M(D) \hookrightarrow M(B)$$

which commutes with $\theta(A)$. In this case the previous results apply and thus given $\alpha = [\varphi, \theta_\infty]$, the Cuntz pairs $(u_t\varphi(\cdot)u_t^*, \theta_\infty(\cdot))$ give a homotopy from (φ, θ_∞) to $(\phi \oplus \theta \oplus \theta \oplus \ldots, \theta \oplus \theta \oplus \ldots) \sim (\phi, \theta) \sim \phi$. $\square$

Uniqueness Part: Prove that the lift is unique up to a particular equivalence relation.

Again we assume A is separable and B is σ-unital. Define

$$B_{as} := C_b([0,\infty), B)/C_0([0,\infty), B)$$

which is called the asymptotic corona algebra or the "path algebra".

Definition. *A $*$-homomorphism $\phi : A \to B$ is called strongly $\mathcal{O}_\infty$-stable if there is a unital embedding*

$$\mathcal{O}_\infty \hookrightarrow \frac{B_{as} \cap \phi(A)'}{Ann\phi(A)}$$

where $Ann\phi(A) = \{x \in B_{as} : x\phi(A) + \phi(A)x = \{0\}\}$.

So ϕ is strongly $\mathcal{O}_\infty$-stable if and only if there exist $s_1, s_2, \ldots : [0,\infty) \to B$ continuous, bounded maps such that

- $\|[s_i(t), \phi(a)]\| \to 0$ as $t \to \infty$, for all $a \in A$;
- $\|(s_i^*(t)s_j(t) - \delta_{ij})\phi(a)\| \to 0$ as $t \to \infty$, for all $a \in A$.

Proposition. *If $A \cong A \otimes \mathcal{O}_\infty$ or $B \cong B \otimes \mathcal{O}_\infty$ then any $*$-homomorphism $\phi : A \to B$ is strongly $\mathcal{O}_\infty$-stable.*

Remark. $A \cong A \otimes \mathcal{O}_\infty$ iff id_A is strongly $\mathcal{O}_\infty$-stable. The word "strongly" comes from the fact, that there is a related notion, that ϕ is $\mathcal{O}_\infty$-stable if the obvious sequential analogue is satisfied.

Definition. *We say that two $*$-homomorphisms $\phi, \psi : A \to B$ are asymptotically Murray von-Neumann equivalent ($\phi \sim_{asMvN} \psi$) if there is a continuous, bounded path $(v_t)_{t\in[0,\infty)}$ in $M(B)$ such that*

- $\|v_t^*\phi(a)v_t - \psi(a)\| \to 0$ *for every $a \in A$;*
- $\|v_t\psi(a)v_t^* - \phi(a)\| \to 0$ *for every $a \in A$.*

If the v_t's can be chosen unitary, then we say that ϕ and ψ are asymptotically unitary equivalent ($\phi \sim_{asu} \psi$).

Fact: $\phi \sim_{asu} \psi \Rightarrow \psi \sim_{asMvN} \psi \Rightarrow KK(\phi) = KK(\psi)$. In general the other directions are not true. However,

Proposition. *If A, B, ϕ, ψ are unital or if B is stable then*

$$\phi \sim_{asu} \psi \Leftrightarrow \phi \sim_{asMvN} \psi.$$

Proposition. *Suppose that A is separable and nuclear, and that $\phi, \theta : A \to B$ are full $*$-homomorphisms. Then,*

(1) If θ is strongly $\mathcal{O}_2$-stable[2] *then $\phi \oplus \theta : A \to M_2(B)$ is strongly $\mathcal{O}_\infty$-stable.*
(2) If ϕ is strongly $\mathcal{O}_\infty$-stable , θ is strongly $\mathcal{O}_2$-stable, then $\phi \oplus \theta \sim_{asMvN} \phi \oplus 0 : A \to M_2(B)$.

Remark: We may replace ϕ with $\phi \oplus \theta$ in the end of the proof of existence. Note that θ factors through $\mathcal{O}_2$, hence it is strongly $\mathcal{O}_2$-stable. Thus by the above proposition, we may assume that ϕ in the existence theorem is strongly $\mathcal{O}_\infty$.

Now we state the theorem of the uniqueness.

Theorem. *Let A be a separable, nuclear C^*-algebra, B be a σ-unital C^*-algebra with a full properly infinite projection. Let also $\phi, \psi : A \to B$ be full, strongly $\mathcal{O}_\infty$-stable $*$-homomorphisms. Then*

$$\phi \sim_{asMvN} \psi \Leftrightarrow KK(\phi) = KK(\psi).$$

For the proof we are going to use a theorem of Dadarlat and Eilers.

Theorem. *Let A be a separable, nuclear C^*-algebra and B be a σ-unital and stable C^*-algebra. Let also $\phi, \psi, \theta : A \to B$ be $*$-homomorphisms such that θ is full and $KK(\phi) = KK(\psi)$. Then there exists a continuous unitary path $(w_t)_{t \in [0,\infty)}$ in $(B \otimes \mathbb{K})^\sim$ such that*

$$\|w_t^*(\phi(a) \oplus \theta_\infty(a))w_t - \psi(a) \oplus \theta_\infty(a)\| \to 0$$

as $t \to \infty$, for every $a \in A$.

Proof of uniqueness: W.l.o.g we can assume that B is stable since $\phi \sim_{asMvN} \psi \Leftrightarrow \phi \otimes e_{11} \sim_{asMvN} \psi \otimes e_{11}$ as maps from A to $B \otimes \mathbb{K}$.
Pick $\theta : A \to B$ as in the proof of existence and from lemma 3) there exists a unitary path $\{u_t'\} \subset M(B \otimes \mathbb{K}_2) \cap \theta_\infty(A)'$ where $\mathbb{K}_2 = \mathbb{K}(\ell^2(\mathbb{N} \setminus \{1\}))$. Let $u_t = 1 \oplus u_t'$. From the above theorem there is a unitary path $(w_t) \in (B \otimes \mathbb{K})^\sim$ and from lemma 2) we have that

$$p_t = u_t^* \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix} u_t$$

[2] It is defined exactly the same way with strongly $\mathcal{O}_\infty$-stable, but with $\mathcal{O}_2$ instead of $\mathcal{O}_\infty$.

is an approximate identity for $B \otimes \mathbb{K}$. If we speed up the u_t's then $\|[p_t, w_t]\| \to 0$. Let

$$v_t = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix} u_t w_t u_t^* \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}$$

and a straightforward computation yields $\phi \oplus \theta \sim_{asMvN} \psi \oplus \theta$ via the v_t's. Now, ϕ, ψ are strongly $\mathcal{O}_\infty$-stable and hence from the last proposition we equivalently get that $\phi \oplus 0 \sim_{asMvN} \psi \oplus 0$. $\square$

An immediate corollary of the existence and uniqueness theorems.

Corollary. *Let A, B be as above. Then*

$$\frac{Hom^{full}_{str.\mathcal{O}_\infty}(A,B)}{\sim_{asMvN}} \xrightarrow{\cong} KK(A,B).$$

If B is stable then we mod out by $\sim_{asu}$.

Proof of Kirchberg-Phillips Theorem: Let A, B be two Kirchberg algebras. Let $\alpha \in KK(A,B)$ and $\beta \in KK(B,A)$ implement the KK-equivalence; $\alpha\beta = 1_A$ and $\beta\alpha = 1_B$. Then from the existence theorem there are $\phi : A \to B$ and $\psi : B \to A$ full, strongly $\mathcal{O}_\infty$-stable $*$-homomorphisms such that $KK(\phi) = \alpha$ and $KK(\psi) = \beta$. Then

$$KK(\phi \circ \psi) = \beta\alpha = KK(id_B)$$
$$KK(\psi \circ \phi) = \alpha\beta = KK(id_A).$$

From the uniqueness theorem we get that $\phi \circ \psi \sim_{asMvN} id_B$ and $\psi \circ \phi \sim_{asMvN} id_A$. Tensoring with $\mathbb{K}$ we get asymptotic unitary equivalence and hence intertwining gives classification. $\square$

4. Approaching the UCT via crossed products / Szabó

Major Problem: Do all separable nuclear C^*-algebras satisfy the UCT?

Theorem. *The following are equivalent:*

(1) *Every separable nuclear C^*-algebra satisfies the UCT;*
(2) *For $p = 2, 3$ and every action $\alpha : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$, the associated crossed product $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_p$ satisfies the UCT;*
(3) *For $p = 2, 3$ and every action $\alpha : \mathbb{Z}_p \curvearrowright W$, the associated crossed product $W \rtimes_\alpha \mathbb{Z}_p$ satisfies the UCT;*

where W is the Razak-Jacelon algebra.

Remark. This also implies that finite group actions on C^*-algebras are difficult to study.

Key fact: For $p = 2, 3$ there exists an action $\alpha : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ such that $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_p \sim_{KK} M_{p^\infty}^{\oplus(p-1)}$. We are going to prove it shortly after we see its usage to prove the theorem.

We are also going to use the fact that any separable, nuclear C^*-algebra A is KK-equivalent to a unital Kirchberg algebra B. This result is due to Kirchberg and for completeness we sketch a proof. Assume that A is unital by adding a unit, if it is necessary. Note that this procedure does not affect if A remains in the UCT class or not. We can assume that $A \cong A \otimes \mathcal{O}_\infty^{st}$ where $\mathcal{O}_\infty^{st} = p\mathcal{O}_\infty p$ for some projection $0 \neq p \in \mathcal{O}_\infty$ with $[p] = 0$ in $K_0(\mathcal{O}_\infty)$. If this is not the case, we can stabilise with $\mathcal{O}_\infty^{st}$ since this does not change the KK-equivalence class. If $A \cong A \otimes \mathcal{O}_\infty^{st}$, then there is a natural embedding $\iota : \mathcal{O}_2 \to A$. Let us explicitely choose two isometries $s_1, s_2 \in A$ with $1 = s_1 s_1^* + s_2 s_2^*$. By Kirchberg's embedding theorem, there exists a natural embedding $k : A \to \mathcal{O}_2$. Now we define an endomorphism $\phi : A \to A$ via

$$\phi(x) = s_1 x s_1^* + s_2 (\iota \circ k)(x) s_2^*.$$

Consider the inductive limit $B = \varinjlim\{A, \phi\}$, and then B is again separable, unital, nuclear and $\mathcal{O}_\infty$-absorbing. Since for all $x \neq 0$, the element

$$s_2 \phi(x) s_2^* = \iota \circ k(x)$$

is the image of the full element $k(x) \in \mathcal{O}_2$, it follows that $\phi(x)$ is also full. Hence B is simple. Finally, it is clear that $KK(\phi) = 1 + KK(\iota \circ k) = 1$ since $\iota \circ k$ factors through $\mathcal{O}_2$. In particular, the connecting maps of this inductive system induce KK-equivalences. Consequently, the canonical embedding $\phi_\infty : A \to B$ induces a KK-equivalence. This can be derived by using Milnor's $\varprojlim^1$-exact sequence for the functor $KK(\cdot, B)$ for the inductive system $\{A, \phi\}$.

Proposition. *Every separable nuclear C^*-algebra satisfies the UCT if and only if every unital Kirchberg algebra satisfies the UCT.*

Lemma. *Let A be a separable C^*-algebra. If both $A \otimes M_{2^\infty}$ and $A \otimes M_{3^\infty}$ satisfy the UCT, then so does A.*

Proof. Consider

$$Z_{2^\infty,3^\infty} = \{f \in C([0,1], M_{2^\infty} \otimes M_{3^\infty}) : f(0) \in M_{2^\infty} \otimes 1, f(1) \in 1 \otimes M_{3^\infty}\}.$$

Then one obtains the following exact sequence:

$$0 \to SM_{6^\infty} \otimes A \to Z_{2^\infty,3^\infty} \otimes A \xrightarrow{ev_0 \oplus ev_1} (M_{2^\infty} \oplus M_{3^\infty}) \otimes A \to 0.$$

Observe that the inclusion $\mathbb{C} \hookrightarrow Z_{2^\infty,3^\infty}$ induces a KK-equivalence. Hence $A \sim_{KK} Z_{2^\infty,3^\infty} \otimes A$. Since $A \otimes M_{2^\infty}$ and $A \otimes M_{3^\infty}$ are in the UCT class then $SM_{6^\infty} \otimes A$, lies in there too. As a result, $Z_{2^\infty,3^\infty} \otimes A$ is in the UCT class and hence the same holds for A. □

Proof. (theorem) We are going to prove the 2) ⇒ 1) direction by contradiction. From the above Proposition we can assume that the UCT fails for a unital Kirchberg algebra A. From the Lemma we get that the UCT fails for $A \otimes M_{p^\infty}^{\oplus(p-1)}$, for p equals 2 or 3. Choose a model action $\alpha : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ as in the "key fact". Then

$$\begin{aligned} A \otimes M_{p^\infty}^{\oplus(p-1)} &\sim_{KK} A \otimes (\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_p) \\ &\cong (A \otimes \mathcal{O}_2) \rtimes_{id \otimes \alpha} \mathbb{Z}_p \\ &\cong \mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_p \end{aligned}$$

which clearly contradicts the assumption. The case for the W algebra uses an analogous "key fact"; if the UCT fails, then it fails for a simple TAF C^*-algebra A with unique trace such that $A \otimes W \cong W$. □

Proof. (key fact) We do the proof for $p = 2$. Set $B = M_{2^\infty} \otimes O_\infty^{st}$. Then $K_0(B) \cong \mathbb{Z}[\frac{1}{2}]$ and B is a Kirchberg algebra. We will express B as an inductive limit in the following way:

Notation: From Robert's classification theorem there exists some $\beta \in Aut(B)$ such that $K_0(\beta) = -id$. Let $B^{(n)} = M_{2^{n-1}} \otimes B$ and $\beta^{(n)} = id_{M_{2^{n-1}}} \otimes \beta$. Consider the following connecting maps

$$\varphi_n : B^{(n)} \oplus B^{(n)} \to B^{(n+1)} \oplus B^{(n+1)},$$

$$\varphi_n(x_0 \oplus x_1) = \begin{pmatrix} x_0 & \\ & \beta^{(n)}(x_1) \end{pmatrix} \oplus \begin{pmatrix} x_1 & \\ & \beta^{(n)}(x_0) \end{pmatrix}.$$

Then it is easy to see that $\varinjlim\{B^{(n)} \oplus B^{(n)}, \phi_n\}$ is a unital, UCT, Kirchberg algebra. Note that simplicity comes from the fact that ϕ_n's are full $*$-homomorphisms. As for the K-theory,

$$\begin{array}{ccc} K_0(B^{(n)} \oplus B^{(n)}) & \xrightarrow{K_0(\phi_n)} & K_0(B^{(n+1)} \oplus B^{(n+1)}) \\ \downarrow & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & \downarrow \\ \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}] & \longrightarrow & \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}] \end{array}$$

Since $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^2 = 2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $\mathbb{Z}[\frac{1}{2}]$ is uniquely 2-divisible we get an isomorphism $K_0(B_\infty) \cong \mathbb{Z}[\frac{1}{2}]$. Therefore, by classification, we have $B_\infty \cong B$.

Now define an action $\gamma : \mathbb{Z}_2 \curvearrowright B$ by flipping the direct sums on each building block. This induces a well defined action on B_∞. We compute $K_0(\phi_n) \circ K_0(\gamma) = -K_0(\phi_n)$ and hence $K_0(\gamma) = -id$. The action γ has the Rokhlin property:

Fact. *Let G be a finite group and A is $M_{|G|^\infty}$-absorbing C^*-algebra. Let $\alpha : G \hookrightarrow A$ be a Rokhlin-action. Then*

$$K_i(A \rtimes_\alpha G) \simeq \bigcap_{g \in G} \ker(id - K_i(\alpha_g))$$

Therefore $K_0(B \rtimes_\gamma \mathbb{Z}_2) \simeq \ker(0) \cap \ker(2) = 0$ and by classification $B \rtimes_\gamma \mathbb{Z}_2 \cong \mathcal{O}_2$.

Then by Takesaki-Takai duality,

$$\begin{aligned} \mathcal{O}_2 \rtimes_{\widehat{\gamma}} \mathbb{Z}_2 &\cong (B \rtimes_\gamma \mathbb{Z}_2) \rtimes_{\widehat{\gamma}} \mathbb{Z}_2 \\ &\cong M_2 \otimes B \\ &= B \\ &\sim_{KK} M_{2^\infty} \end{aligned}$$

□

5. Tracial Rokhlin Property for Conditional Expectations / Lee

For this talk all C^*-algebras are separable and unital.

Conditional expectation:
Let $P \subset A$ be the inclusion of a unital C^*-algebra with a conditional expectation $E : A \to P$.

- A quasi-basis for E is a finite set $\{(u_i, v_i)\} \subset A \times A$ such that every $a \in A$ can be written as $a = \sum u_i E(v_i a) = \sum E(au_i)v_i$.
- When $\{(u_i, v_i)\}_{i=1}^n$ is a quasi-basis for E we define
$$IndE = \sum_{i=1}^{n} u_i v_i.$$
In this case $IndE < \infty$. If there is no quasi-basis then $IndE = \infty$.

Remark:

- $IndE$ is central in A.
- If we know that $IndE < \infty$, then we can choose $\{(u_i, u_i^*)\} \subset A \times A$.
- $IndE \in A^+$.
- $IndE$ is invertible.

Basic construction of a pre-Hilbert P-module:
Using the conditional expectation E we turn A into a pre-Hilbert P-module by setting $< a_1, a_2 >= E(a_1^* a_2)$ and denote by $\mathcal{E}_E$ the completion with respect to the inner product. Let $\eta_E : A \hookrightarrow \mathcal{E}_E$ be the inclusion, then the Jones projection $e_p \in \mathcal{L}(\mathcal{E}_E)$ is defined by
$$e_p(\eta_E(a)) = \eta_E(E(a)).$$
We also have a left action of A on $\mathcal{E}_E$ given by $\lambda : A \to \mathcal{L}(\mathcal{E}_E)$ where $\lambda(a)(\eta_E(x)) = \eta_E(ax)$. Define $C_r^* < A, e_p >= \overline{span\{xe_p y : x, y \in A\}} = K(\mathcal{E}_E)$.

Proposition. *E is of finite index if and only if $C_r^* < A, e_p >$ has an identity and there exists $c > 0$ such that $E(x^*x) \geq cx^*x$ for $x \in A$.*

Note: There is a maximal construction $C_{max}^* < A, e_p >$ which is equal to $C_r^* < A, e_p >$ if $IndE < \infty$. In that case we will just write $C^* < A, e_p >$.

Dual conditional expectation:
Let $\hat{E} : C^* < A, e_p > \to A$ be given by $xe_p y \mapsto (IndE)^{-1}xy$. If $\{(u_i, u_i^*)\}$ is a quasi-basis for E, then $\{(u_i e_p \sqrt{IndE}, (u_i e_p \sqrt{IndE})^*)\}$ is a quasi-basis for $\hat{E}$.

Example. *Let G be a finite group and $\alpha : G \curvearrowright A$ a saturated (outer) action. Let $E : A \to A^\alpha$: the fixed point algebra, that is given by $E(a) =$*

$\frac{1}{|G|}\sum_{g\in G}\alpha_g(a)$. *Then* $A\rtimes_\alpha G\cong C^* < A, e_p >$ *and* $A\rtimes_\alpha G\to A$ *has* $\hat{E}$ *as conditional expectation.*

Izumi 04'

Definition. *Let* G *be a finite group. We say that* $\alpha : G \curvearrowright A$ *has the Rokhlin property if there is a partition of unity* $\{e_g\}_{g\in G}$*: projections in* $A_\infty\cap A'$*, such that* $\alpha_{h,\infty}(e_g) = e_{hg}$*, where* α_∞ *is the induced action* $G\curvearrowright A_\infty$*.*

Definition. *Let* G *be a finite group. Then* $\alpha : G \curvearrowright A$ *is approximately representable if there exists* $U : G \to (A^\alpha)_\infty$ *such that* $\alpha_g(x) = u_g x u_g^*$ *in* A_∞*.*

The next one is a duality result.

Proposition. *Let* G *be a finite abelian group acting on* A*. We denote by* $\hat{\alpha}$ *the dual action of* α *on* $A\rtimes_\alpha G$*. Then*

1. *the action* α *has the Rokhlin property if and only if the dual action* $\hat{\alpha}$ *is approximately representable.*
2. *the action* α *is approximately representable if and only if the dual action* $\hat{\alpha}$ *has the Rokhlin property.*

Definition. *A conditional expectation* $E : A\to P$ *is said to have the Rokhlin property if there exists a projection* $e\in A_\infty\cap A'$ *such that* $E_\infty(e) = (IndE)^{-1}$ *and the map* $x\mapsto xe$*, for* $x\in A$*, is injective.*

Definition. *A conditional expectation* $E : A \to P$ *is said to be approximately representable if there exists a projection* $e\in P_\infty\cap P'$ *and a set* $\{(u_i, u_i^*)\}\subset A\times A$ *such that*

1. $exe = E(x)e$*;*
2. $\sum u_i e u_i^* = 1$*;*
3. *the map* $x\mapsto xe$*, for* $x\in P$*.*

We get the following duality result.

Proposition. *Let* $E : A\to P$ *be a conditional expectation with* $IndE < \infty$*. Then*

1. *The conditional expectation* E *has the Rokhlin property if and only if the dual* $\hat{E}$ *is approximately representable.*
2. *The conditional expectation* E *is approximately representable if and only if the dual* $\hat{E}$ *has the Rokhlin property.*

Chris Phillips 11', tracial

Definition. *Let* A *be also simple, infinite dimensional, and let* $\alpha : G\curvearrowright A$ *be the action of a finite group* G*. We say that* α *has the tracial Rokhlin property if for every* $z\in A_\infty^+$ *there exist mutually orthogonal projections* e_g *in* $A_\infty\cap A'$ *such that*

(1) $\alpha_{h,\infty}(e_g) = e_{hg}$;
(2) $1 - \sum_{g \in G} e_g$ *is Murray-von Neumann equivalent to a projection in* $\overline{zA_\infty z}$.

Definition. *Let* G *a finite abelian group that acts on* A *by* α. *We say that* α *is tracially approximately representable if for every* $0 \neq z \in A_\infty^+$ *there exists a projection* $e \in A_\infty \cap A'$ *and a unitary representation* $w : G \to eA_\infty e$ *such that*

(1) $\alpha_g(eae) = w_g(eae)w_g^*$;
(2) $\alpha_{g,\infty}(w_h) = w_h$;
(3) $1 - e$ *is Murray-von Neumann equivalent to a projection in* $\overline{zAz}$.

The analogous duality result is the following.

Proposition. *Let* G *be a finite abelian group that acts on* A *by* α. *Then*

(1) *The action* α *has the tracial Rokhlin property if and only if the dual action* $\hat{\alpha}$ *is tracially approximately representable.*
(2) *The action* α *is tracially approximately representable if and only if the dual action* $\hat{\alpha}$ *is has the tracial Rokhlin property.*

Definition. *Let* $E : A \to P$ *be a conditional expectation. We say that it has the tracial Rokhlin property if for every* $0 \neq z \in A_\infty^+$, *there exists a projection* $e \in A_\infty \cap A'$ *such that*

(1) $E_\infty(e)IndE = g$ *a projection;*
(2) $1 - g$ *is Murray-von Neumann equivalent to a projection in* $\overline{zA_\infty z}$;
(3) *the map* $x \mapsto xe$, *for* $x \in A$, *is injective.*

Definition. *Let* $E : A \to P$ *be a conditional expectation. We say that it is tracially approximately representable if for every* $0 \neq z \in A_\infty^+$, *there exists a projection* $e \in P_\infty \cap P'$, *some* $r \in A_\infty \cap A'$, *and a finite set* $\{(u_i, u_i^*)\}$ *such that*

(1) $exe = E(x)e$;
(2) $\sum u_i e u_i^* = r$;
(3) $re = e = er$;
(4) $1 - r$ *is Murray-von Neumann equivalent to a projection in* $\overline{zA_\infty z}$.

Finally, there is an analogous result.

Proposition. *Let* $E : A \to P$ *be a conditional expectation with finite depth. Then*

(1) *The conditional expectation* E *has the tracial Rokhlin property if and only if the dual* $\hat{E}$ *is tracially approximately representable.*
(2) *The conditional expectation* E *is tracially approximately representable if and only if the dual* $\hat{E}$ *has the tracial Rokhlin property.*

6. Cartan subalgebras, automorphisms and the UCT problem / Barlak

Question(UCT Problem) Does every separable, nuclear C^*-algebra satisfy the UCT?

A classical result by Kirchberg reduces the UCT problem to Kirchberg algebras:

Theorem. *Every separable, nuclear C^*-algebra is KK-equivalent to a unital Kirchberg algebra.*

In Gabor's lecture it was shown that this theorem can be used to reduce the UCT problem to crossed product of the Cuntz algebra $\mathcal{O}_2$ by finite group actions:

Theorem. *The following are equivalent:*

1) Every separable, nuclear C^*-algebra satisfies the UCT;

2) For $p = 2, 3$ and for every outer strongly approximately inner action $\alpha : \mathbb{Z}_p \hookrightarrow \mathcal{O}_2$, the crossed product $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_p$ satisfies the UCT.

This reduction of the UCT problem uses that many Kirchberg algebras (absorbing suitable UHF algebras) can be written as crossed products of $\mathcal{O}_2$ by actions of $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Here $\alpha : \mathbb{Z}_p \hookrightarrow \mathcal{O}_2$ is said to be strogly approximately inner if

$$\alpha = \lim_{n\to\infty} Ad(u_n)$$

for some unitaries $u_n \in \mathcal{O}_2^\alpha$. This is equivalent to the dual action $\widehat{\alpha} : \mathbb{Z}_p \hookrightarrow \mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_p$ having the Rokhlin property.

By employing the rigid nature of the Rokhlin property, strongly approximately inner actions of $\mathcal{O}_2$ are particularly nice to work with. In fact, Izumi successfully classified such actions on $\mathcal{O}_2$ in terms of their crossed products (and some additional information on their dual actions).

It is an open question whether all outer actions of $\mathbb{Z}_p$ on $\mathcal{O}_2$ are strongly approximately inner.

One may now ask the following:

Question. *Let $\alpha : \mathbb{Z}_p \hookrightarrow \mathcal{O}_2$ be an outer strongly approximately action. What can we say about α if we assume that $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_p$ satisfies the UCT?*

We will present a structure result for such actions, which involves Renault's notion of a Cartan subalgebra:

Definition. *A C^*-subalgebra B of a C^*-algebra A is called a Cartan subalgebra if*

1) B contains an approximate unit for A;

2) B is a maximal abelian $*$-subalgebra;

3) $C^*(\{a \in A : aBa^* \subset B \text{ and } a^*Ba \subset B\}) = A$;

4) There exists a faithful conditional expectation $A \to B$.

(A, B) is called then a Cartan pair.

We give several examples of Cartan pairs:

Example. *1)* $(M_n(\mathbb{C}), \{\textit{Diagonal matrices in } M_n(\mathbb{C})\})$.

2) $(C_0(X) \rtimes_{\alpha,r} G, C_0(X))$ *with* X *a locally compact Hausdorff space and* α *a topologically free discrete group action;*

3) $(\mathcal{O}_n, \mathcal{D}_n)$ *where* $\mathcal{D}_n$ *is the abelian* C^**-subalgebra generated by all range projections* $S_\alpha S_\alpha^*$*, where* α *is a finite word in* $\{1, \ldots, n\}$.

Many simple, nuclear C^*-algebras that are classifiable (in the sense of the Elliott program) have Cartan subalgebras.

Renault (2008) has shown that for any Cartan pair (A, B) with A separable, there exists a twisted etale, locally compact, Hausdorff groupoid (G, Σ) such that

$$(A, B) \simeq (C^*_{red}(G, \Sigma), C_0(G^{(0)}).$$

Theorem. *Let* A *be a separable, nuclear* C^**-algebra. If* A *has a Cartan subalgebra, then* A *satisfies the UCT.*

For C^*-algebras associated with etale, locally compact, Hausdorff groupoids this is due to a remarkable result of Tu (1999). We basically adapted his results and techniques to the setting of twisted groupoid C^*-algebras.

Remark. *By work of Spielberg (2007) or Katsure (2008) and Yeend (2006 + 2007), every UCT Kirchberg algebra has a Cartan subalgebra.*

Combining this with our result and Kirchberg's reduction of the UCT problem to Kirchberg algebras, the UCT problem turns out to have an affirmative answer if every Kirchberg algebra has a Cartan subalgebra.

A family $\mathcal{S} \subset A$ of partial isometries in a C^*-algebra is called an inverse semigroup if it is closed under multiplication and the $*$-operation. Let

$$E(\mathcal{S}) = \{e \in \mathcal{S} : e = e^2\} = \{e \in \mathcal{S} : e = e^2 = e^*\}$$

denote the semi-lattice of idempotent elements and write $C^*(E(\mathcal{S}))$ for the commutative C^*-subalgebra of A generated by $E(\mathcal{S})$.

Theorem. *Let* $n = p^k$ *for some prime number* p *and some* $k \geq 1$*. Let* $\alpha : \mathbb{Z}_n \hookrightarrow \mathcal{O}_2$ *be an outer strongly approximately inner action. Then the following are equivalent:*

1) $\mathcal{O} \rtimes_\alpha \mathbb{Z}_n$ *satisfies the UCT;*

2) There exists an inverse semigroup $\mathcal{S} \subset \mathcal{O}_2$ *of* α*-homogeneous partial isometries such that* $\mathcal{O}_2 = C^*(\mathcal{S})$ *and* $C^*(E(\mathcal{S}))$ *is a Cartan subalgebra in both* $\mathcal{O}_2^\alpha$ *and* $\mathcal{O}_2$ *(with spectrum homeomorphic to the Cantor set);*

3) There exists a Cartan subalgebra $C \subset A$ *such that* $\alpha(C) = C$*.*

Proof. 2) $\to$ 3) is trivial.

3) $\to$ 2): Let $\alpha : \mathbb{Z}_n \hookrightarrow \mathcal{O}_2$ be outer strongly approximately inner such that $\alpha(C) = C$ for some Cartan subalgebra $C \subset \mathcal{O}_2$. We have to show that $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_n$ satisfies the UCT. Using Renault's characterisation of Cartan pairs, we may identify

$$(A, C) \simeq (C^*_r(G, \Sigma), C(G^{(0)}))$$

for a suitable twisted groupoid (G,Σ). Going through the construction of (G,Σ) and using that $\alpha(C)=C$, one can see that under this identification α is induced from a twisted groupoid automorphism. This gives rise to a twisted semi-direct groupoid $(\mathbb{Z}_n \ltimes G, \mathbb{Z}_n \ltimes \Sigma)$, One can show that $A \rtimes_\alpha \mathbb{Z}_n \simeq C^*_r(G \ltimes \mathbb{Z}_n, \Sigma \ltimes \mathbb{Z}_n)$, from which the UCT for $A \rtimes_\alpha \mathbb{Z}_n$ can be deduced.

1) $\to$ 2): Let $\alpha : \mathbb{Z}_n \curvearrowright \mathcal{O}_2$ be outer strongly approximately inner such that $A := \mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_n$ satisfies the UCT. Using the Pimsner-Voiculescu sequence for $\mathbb{Z}_n$-actions, that n is a prime power and Kirchberg-Phillips classification, one checks that $A \simeq A \otimes M_{n^\infty}$.

By a result of Izumi, the dual action $\widehat{\alpha} = \gamma : \mathbb{Z}_n \curvearrowright A$ has the Rokhlin property. Combining results of Katsura and Spielberg, we find an action $\beta : \mathbb{Z}_n \curvearrowright A$ and an inverse semigroup $\widetilde{\mathcal{S}} \subset A$ of partial isometries such that:

$$K_*(\beta) = K_*(\gamma),$$

$$C^*(\widetilde{\mathcal{S}}) = A \text{ and } C^*(E(\widetilde{\mathcal{S}})) \subset A \text{ is a Cartan subalgebra },$$

$$\beta(C^*(E(\widetilde{\mathcal{S}}))) = C^*(E(\widetilde{\mathcal{S}})).$$

Using a model action result of B.-Szabo and Izumi's rigidity result for actions with Rokhlin property, we get that

$$(A,\gamma) \simeq \lim_{k\to\infty} ((C(\mathbb{Z}_n) \otimes M_n^{\otimes k-1} \otimes A, \varphi_k), \mathbb{Z}_n\text{-shift} \otimes id_{M_n^{\times k-1} \otimes A},$$

where

$$\varphi_k(f)(m) = \sum_{l=0}^{n-1} e_{l,l} \otimes (id_{M_n^{\otimes k-1}} \otimes \beta^l)(f(m+l)).$$

Here we use the fact that A absorbs M_{n^∞} tensorially.

Let $B \subset A$ be the abelian C^*-subalgebra corresponding to

$$\lim_{k\to\infty} (C(\mathbb{Z}_n \otimes D_n^{\otimes k-1} \otimes C^*(E(\widetilde{\mathcal{S}})), \varphi_k).$$

It holds that $\gamma(B)=B$. Using the explicit inductive limit construction, one can show that $B \subset A$ is indeed a Cartan subalgebra. Similarly, by passing to crossed products, one shows that $B \subset A \rtimes_\gamma \mathbb{Z}_n$ is Cartan subalgebra as well. ($A \rtimes_\gamma \mathbb{Z}_n = \mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_n \rtimes_{\widehat{\alpha}} \mathbb{Z}_n \simeq \mathcal{O}_2$ by Takai duality). Not that B is fixed by $\widehat{\gamma} = \widehat{\widehat{\alpha}}$ point-wise.

Let $\mathcal{S} \subset A \rtimes_\gamma \mathbb{Z}_n$ denote the inverse semigroup generated by all partial isometries $s \in A$ with $sBs^* + s^*Bs \subset B$ and the canonical unitary $u \in A \rtimes_\gamma \mathbb{Z}_n$ implementing γ. Then $\mathcal{S}$ is homogeneous for $\widehat{\gamma}$, $C^*(\mathcal{S}) = A \rtimes_\gamma \mathbb{Z}_n$ and $C^*(E(\mathcal{S})) = B$. This yields the assertion for $\widehat{\widehat{\alpha}}$. Employing Takai dualitym one can deduce from this with some extra work the assertion for α as well. This shows 2. □

Using this characterisation of the UCT for certain crossed products of $\mathcal{O}_2$ and the reduction of the UCT problem from the beginning of the talk, one can deduce the following further characterisation of the UCT problem:

Theorem. *The following statements are equivalent:*

1) Every separable, nuclear C^*-algebra satisfies the UCT;

2) For every prime number $p \geq 2$ and every outer strongly approximately inner action $\alpha : \mathbb{Z}_p \hookrightarrow \mathcal{O}_2$ there exists an inverse semigroup $\mathcal{S} \subset \mathcal{O}_2$ of α-homorgeneous partial isometries such that $\mathcal{O}_2 = C^*(\mathcal{S})$ and $C^*(E(\mathcal{S}))$ is a Cartan subalgebra in both $\mathcal{O}_2^\alpha$ and $\mathcal{O}_2$ (with spectrum homeomorphic to the Cantor set);

3) Every outer strongly approximately inner $\mathbb{Z}_p$-action on $\mathcal{O}_2$ with $p = 2$ or $p = 3$ fixes some Cartan subalgebra $C \subset \mathcal{O}_2$ globally.

A sufficient condition for an affirmative answer to the UCT problem can also be formulated in terms of aperiodic automorphisms of $\mathcal{O}_2$.

Here $\alpha \in Aut(\mathcal{O}_2)$ is said to be aperiodic if α^n is outer for all $0 \neq n \in \mathbb{Z}$.

Let us first recall Nakamura's classification of aperiodic automorphisms on Kirchberg algebras.

Theorem. *Let A be a unital Kirchberg algebra and let $\alpha, \beta \in Aut(A)$ be two aperiodic automorphisms. Then the following assertions are equivalent:*

1) $KK(\alpha) = KK(\beta)$;

2) α and β are cocycle conjugate via an automorphisms with trivial KK-class, that is, there exists $\mu \in Aut(A)$ with $KK(\mu) = 1_A$ and $u \in \mathcal{U}(\mathcal{O}_2)$ such that $Ad(u)\alpha = \mu\beta\mu^{-1}$.

In particular, all aperiodic automorphisms of $\mathcal{O}_2$ are cocycle conjugate to each other.

Proposition. *The UCT problem has an affirmative answer if for every aperiodic automorphism $\alpha \in Aut(\mathcal{O}_2)$ there exists a Cartan subalgebra $B \subset \mathcal{O}_2$ such that $\alpha(B) = B$.*

Observe that for each aperiodic automorphism $\alpha \in Aut(\mathcal{O}_2)$ it holds that $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z} \simeq \mathcal{O}_2$.

Question. *Is the converse of the above statement true as well? In other words, are the following two statements equivalent:*

1) Every separable, nuclear C^*-algebra satisfies the UCT;

2) For every aperiodic automorphism $\alpha \in Aut(\mathcal{O}_2)$ there exists a Cartan subalgebra $B \subset \mathcal{O}_2$ such that $\alpha(B) = B$?

We will give an idea of the proof of the last proposition:

Proof. Assume that every aperiodic automorphism of $\mathcal{O}_2$ fixes some Cartan subalgebra. Let $m \geq 2$ and let $\alpha : \mathbb{Z}_m \hookrightarrow \mathcal{O}_2$ be an outer action. It suffices to show that $\alpha \otimes id_{\mathcal{O}_\infty} : \mathbb{Z}_m \hookrightarrow \mathcal{O}_2 \otimes \mathcal{O}_\infty \simeq \mathcal{O}_2$ fixed some Cartan subalgebra (globally), as in this case $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_m$ satisfies the UCT.

One can find an aperiodic automorphism $\gamma \in Aut(\mathcal{O}_\infty)$ with the property that there exist natural numbers n_k with $n_k \equiv 1 \mod m$, $k \geq 1$, such that $\lim_{k\to\infty} \gamma^{n_k} = id_{\mathcal{O}_\infty}$ is point-norm topology. Then $\alpha \otimes \gamma \in Aut(\mathcal{O}_2 \otimes \mathcal{O}_\infty)$ is aperiodic and $((\alpha \otimes \gamma)^{n_k})_k$ converges to $\alpha \otimes id_{\mathcal{O}_\infty}$ is point-norm topology. By assumption, we find some Cartan subalgebra $C \subset \mathcal{O}_2 \otimes \mathcal{O}_\infty$ such that

$(\alpha \otimes \gamma)(C) = C$. However, then $(\alpha \otimes \gamma)^{n_k}(C) = C$ for all k and therefore also $(\alpha \otimes id_{\mathcal{O}_\infty})(C) = C$. $\square$

7. Irreducible representations of nilpotent groups generate classifiable C*-algebras / Gillaspy

Outline:

- Proof that $C^*_\pi(G)$ satisfies the UCT when G is f.g nilpotent;
- $C^*_\pi(G) \cong pC^*(G/Z(G), \omega)$;
- Structure of $C^*_\pi(H)$ when H is virtually nilpotent.

Classifiability of $C^*_\pi(G)$:
For now:

- G is a finitely generated nilpotent group;
- π is an irreducible unitary representation of G;
- $C^*_\pi(G)$ is the associated C^*-algebra.

We will show that $C^*_\pi(G)$ satisfies the UCT. This implies [Lan73, MR76, EM15, TWW17] that $C^*_\pi(G)$ is classifiable. Lance proved that $C^*_\pi(G)$ is nuclear; Moore and Rosenberg proved that every primitive ideal in $C^*(G)$ is actually maximal. Eckhardt and McKenney proved that $C^*_\pi(G)$ has finite nuclear dimension.

Nilpotent groups:

Definition. *A group G is nilpotent if the series*

$$G \triangleright [G, G] \triangleright [G, [G, G]] \triangleright \dots$$

terminates in $\{e\}$.

Examples:

- Abelian groups
- finite p-groups
- $UT(n, \mathbb{Z})$

If $M, N \in UT(n, \mathbb{Z})$, then $MN(NM)^{-1} \in UT(n, \mathbb{Z})$ has a zero super-diagonal. If $S \in UT(n, \mathbb{Z})$ has a zero super-diagonal, then $MS(SM)^{-1}$ has two super-diagonals of zeros. Done by induction. □

An example: Let $G = UT(4, \mathbb{Z}), \theta \in (0, 1) \setminus \mathbb{Q}$, Θ the trace on $UT(4, \mathbb{Z})$:

$$\Theta \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{cases} e^{2\pi i c\theta}, & a = b = d = e = f = 0 \\ 0, & else. \end{cases}$$

Write π_Θ for the GNS representation associated to Θ. Every irreducible representation of G is of the form π_Θ; in fact, [CM84] for any nilpotent group G, every irreducible representation is of the form π_τ for some trace τ.

Moreover, [EKM16] $C^*_{\pi_\Theta}(G) \cong C^*(G/Z(G), \omega_\theta) \cong (A_\theta \otimes A_\theta) \rtimes_\beta \mathbb{Z}$. The twist is given by $\omega_\theta(\gamma, \eta) = \Theta(GH(HG)^{-1})$, where G is a representative of γ with $c = 0$.

A quick advertisement of classification:
The isomorphisms $C^*_{\pi_\Theta}(G) \cong C^*(G/Z(G), \omega_\theta) \cong (A_\theta \otimes A_\theta) \rtimes_\beta \mathbb{Z}$ arise from the group structure of G. [EKM16] also found some isomorphisms

$$Ell(C^*_{\pi_\theta}(G)) \cong Ell(C^*_{\pi_{\tilde{\theta}}}(G))$$

which are not evident from the definition of G or $C^*_{\pi_\theta}(G)$! Since the algebras $C^*_{\pi_\theta}(G)$ are classifiable, these isomorphisms give us new perspectives on $C^*_{\pi_\theta}(G)$.

First step:

Theorem (Eckhardt-Gillaspy)**.** *If τ is a trace on a f.g nilpotent group G such that $\tau(x) \neq 0 \Rightarrow x \in G_f$ (x has a finite conjugacy class), then*

$$C^*_{\pi_\tau} \cong C^*_{\pi_\tau}(G_f) \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}$$

satisfies the UCT.

Proof.

- G is f.g nilpotent, hence G/G_f is torsion free and f.g nilpotent. The proof relies on the fact that $T(G)$ is a finite subgroup of G (since G is polycyclic and thus so are all subgroups).
- $G \cong G_f \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}$.
- $G_f/Z(G_f)$ is finite; thus, $C^*(G_f)$ is subhomogeneous.
- Therefore, $C^*_{\pi_\tau}(G_f)$ satisfies the UCT by [RS87]. In fact $C^*_{\pi_\tau}(G_f) \in \mathcal{N}$.

□

The main theorem:
Proof that $C^*_\pi(G)$ satisfies the UCT:

- If π is faithful on $G \subset C^*(G)$ then $\pi = \pi_\tau$ for an extreme trace τ.
- In this case $\tau(G \setminus G_f) = 0$ [CM84] - so apply the above Theorem.
- If π is not faithful on $G \subset C^*(G)$, then replace G by $G/\ker\pi$- also f.g nilpotent. □

Theorem (Eckhardt- Gillaspy)**.** *for G f.g nilpotent and π a faithful irreducible representation of G, there exists an $N \leq Z(G)$ such that*

$$C^*_\pi(G) \cong pC^*(G/N, \sigma)$$

for a 2-cocycle σ which is homotopic to the trivial 2-cocycle.

Proof.

- π is faithful, hence $\pi = \pi_\tau$ for an extreme trace τ.

- $Z(G)$ is f.g abelian, hence there is an $N \leq Z(G)$ torsion free, finite rank.
- Let ω be the restriction of τ to N; then $C^*_{\pi_\tau}(G) \cong pC^*_{\pi_\omega}(G)$ for a central projection p. This relies on:
  - $\pi_\tau \prec \pi_\omega$;
  - Conditional expectations $E_i : C^*_i(G) \to C^*_i(G_f)$ for $i = \pi_\tau, \pi_\omega$.
  - $C^*_{\pi_\omega}(G) \cong C^*(G/N, \sigma)$ where

    $$\sigma(sN, tN) = \omega(c(sN)c(tN)c(stN)^{-1})$$

    for a choice $c : G/N \to G$ of coset representatives [EKM16].
  - Since $N \cong \mathbb{Z}^d$ and $\omega : N \to \mathbb{T}$,

    $$\omega \in \widehat{\mathbb{Z}^d} \cong \mathbb{T}^d$$

    which is path-connected.

□

Structure for virtually nilpotent groups [EGM17]:

Now, let G be a f.g virtually nilpotent group; that is, G has a normal nilpotent subgroup of finite index, such that $G = N \rtimes_{\alpha,\omega} G/N$. Again, π is an irreducible representation of $C^*(G)$. Via, Stinespring, embed $C^*_\pi(G)$ into $C^*_\sigma(N) \otimes M_{G/N}$ for an irreducible σ. G acts on $\ker(\sigma)$ by conjugation. Define $H = stab_G(\ker(\sigma))$.

- $C^*_\pi(G)$ is a direct summand of $(C^*_{\pi_\tau}(N) \rtimes_{\alpha,\omega} H/N) \otimes M_{G/H}$.
- $C^*_{\pi_\tau}(N)$ is a direct sum of simple $\mathcal{Z}$-stable C^*-algebras.
- If H/N is simple, the twisted action (α, ω) must be either strongly outer, or inner.

8. Structure and classification of nuclear C*-algebras: The role of the UCT / Winter

The question whether all separable nuclear C*-algebras satisfy the Universal Coefficient Theorem remains one of the most important open problems in the structure and classification theory of such algebras. It also plays an integral part in the connection between amenability and quasidiagonality. I will discuss several ways of looking at the UCT problem, and phrase a number of intermediate questions. This involves the existence of Cartan MASAs on the one hand, and certain kinds of embedding problems for strongly self-absorbing C*-algebras on the other.

Theorem (A). *[Tikuisis - White - Winter] Let A be a separable, nuclear C^*-algebra that satisfies the UCT. Then every faithful trace $\tau \in T(A)$ is quasidiagonal; that is, there exists an embedding $A \hookrightarrow \mathcal{Q}_\omega$ such that it makes the following diagram commute:*

$$\begin{array}{ccc} A & \hookrightarrow & \mathcal{Q}_\omega \\ & \searrow^{\tau} & \downarrow \tau_{\mathcal{Q}_\omega} \\ & & \mathbb{C} \end{array}$$

Note:

- Gabe extended this for A being exact and τ being amenable.
- Very nice alternative proof by Schafhauser.

Idea of proof

We can construct $\acute{\Lambda}$ so to get the following commuting diagram:

$$\begin{array}{ccc} C_0(0,1]\otimes A & \xrightarrow{\acute{\Lambda}} & \mathcal{Q}_\omega \\ & \searrow^{L\otimes\tau} & \downarrow \tau_{\mathcal{Q}_\omega} \\ & & \mathbb{C} \end{array}$$

where L is the trace given by Lebesgue integration. To see that, consider some $h \in C_0(0,1]$ with $spec(h) = [0,1]$. Then $\acute{\Lambda}(t \otimes 1_A)$ corresponds to h and by a functional calculus argument we define $\acute{\Lambda}$.

We do the same for $C_0[0,1)$ and get an $\grave{\Lambda}$ such that

$$\begin{array}{ccc} C_0[0,1)\otimes A & \xrightarrow{\grave{\Lambda}} & \mathcal{Q}_\omega \\ & \searrow^{L\otimes\tau} & \downarrow \tau_{\mathcal{Q}_\omega} \\ & & \mathbb{C}. \end{array}$$

Now, what we would like is to find a unitary $u \in \mathcal{Q}_\omega$ such that

$$u^*\acute{\Lambda}_{C_0(0,1)\otimes A}u = \grave{\Lambda}_{C_0(0,1)\otimes A} \quad (*)$$

(in other words to glue $\acute{\Lambda}$ and $\grave{\Lambda}$) and then get an embedding $A \hookrightarrow C[0,1]\otimes A \xrightarrow{\rho} \mathcal{Q}_\omega$ where ρ is defined as follows:

Given $f \in C[0,1]$ we can write it as a sum of elements $\grave{f} \in C_0[0,1)$ and $\acute{f} \in C_0(0,1]$, and then $\rho(f\otimes a) = u^*\acute{\Lambda}(\acute{f}\otimes a)u + \grave{\Lambda}(\grave{f}\otimes a)$ which is well-defined because of $(*)$. Next note that conjugation by u does not affect the trace.

However, this is too good to be true! What we do instead is to stabilise.

Theorem (**B**)**.** *[Lin - Dadarlat - Eilers] Assume that we are given $\mathcal{F} \subset\subset C_0(0,1)\otimes A$ finite, $\epsilon > 0$, $\phi, \psi, i : C_0(0,1)\otimes A \to \mathcal{Q}_\omega$ such that $KK(\phi) = KK(\psi)$ and i is "totally full"; for any non-zero element x , $i(x)$ is full. Then there are $N \in \mathbb{N}$ and a unitary $v \in M_{N+1}(\mathcal{Q}_\omega)$ such that*

$$v(\phi \oplus i^{\oplus N})v^* \approx_{\mathcal{F},\epsilon} \psi \oplus i^{\oplus N}.$$

The idea is the following: Stabilise $\grave{\Lambda}$, $\acute{\Lambda}$ and get $\grave{\Lambda}\otimes 1_{M_{2N+1}}$ and $\acute{\Lambda}\otimes 1_{M_{2N+1}}$ respectively. Then write each diagonal matrix as a column and perform the following inductive steps. Start with $\grave{\Lambda}\otimes 1_{M_{2N+1}}$ and next write $\acute{\Lambda}\oplus\grave{\Lambda}\otimes 1_{M_{2N}}$ in a way that we have stabilised $\grave{\Lambda}$ and $\acute{\Lambda}$ by $\grave{\Lambda}\otimes 1_{M_{2N}}$. By the above theorem we get that these two columns are approximately unitary equivalent. Keep doing this until step N and at step $N+1$ change the process and stabilise from top to bottom. In the end we get a sequence of approximate unitary equivalences from $\grave{\Lambda}\otimes 1_{M_{2N+1}}$ to $\acute{\Lambda}\otimes 1_{M_{2N+1}}$. These can be seen from the following figure which is a $(2N+1)\times(2N+1)$ matrix.

$$\begin{array}{cccccc}
\grave{\Lambda} & \acute{\Lambda} & \acute{\Lambda} & \dots & \dots & \acute{\Lambda} \\
\grave{\Lambda} & \grave{\Lambda} & \acute{\Lambda} & \dots & \dots & \acute{\Lambda} \\
\grave{\Lambda} & \grave{\Lambda} & \grave{\Lambda} & \acute{\Lambda} & \dots & \acute{\Lambda} \\
\vdots & \vdots & \vdots & \grave{\Lambda} & & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
\grave{\Lambda} & \grave{\Lambda} & \grave{\Lambda} & \grave{\Lambda} & & \acute{\Lambda}
\end{array}$$

Remark(Rørdam): N depends on $\mathcal{F}, \epsilon$ but also on ϕ, ψ, i. Hence it depends on $\grave{\Lambda}$ and $\acute{\Lambda}$, which depend on N. So we get something circular.

Theorem (**C**)**.** *[Dadarlat - Eilers] Assume that A is in the UCT class. Then we may choose N to depend only on $A, \mathcal{F}, \epsilon$ and the fullness of i.*

Therefore with the UCT assumption the above idea works.

Note: For Theorem A it would suffice to have an embedding

$$KK(C_0(0,1)\otimes A, \prod_{\mathbb{N}} \mathcal{Q}_\omega) \hookrightarrow \prod_{\mathbb{N}} KK(C_0(0,1)\otimes A, \mathcal{Q}_\omega).$$

Now if we assume UCT then we get the embedding.

Classification

Definition. *Let B be a separable, simple, unital C^*-algebra. We call it tracially approximately finite dimensional (TAF) if for every $\mathcal{F} \subset\subset B$ finite, $\epsilon > 0$, and $0 \neq d \in B^1_+$ there exist a finite dimensional $F \subset B$ and $x \in B^1$ such that*

- $1_F a \approx_\epsilon a 1_F \approx_\epsilon 1_F a 1_F \in_\epsilon \mathcal{F}$, *for every* $a \in \mathcal{F}$;
- $x^* x = 1_B - 1_F$;
- $xx^* \in \overline{dBd}$.

Suppose that we have $\phi, \psi : A \to B$ where A is a "suitable" C^*-algebra, B is TAF and nuclear and they "sufficiently agree on K-theory".

Question: Are ϕ and ψ some sort of approximately equivalent (s.o.a.u)? Something like

$$\begin{pmatrix} & & \\ & \phi & \\ & & \end{pmatrix} \approx_{\mathcal{F},\epsilon} \left(\begin{array}{c|c} \phi_{\mathcal{F}} & 0 \\ \hline 0 & \hat{\phi} \end{array}\right) \sim_{s.o.a.u} \left(\begin{array}{c|c} \psi_{\mathcal{F}} & 0 \\ \hline 0 & \hat{\phi} \end{array}\right)$$

and then

$$\left(\begin{array}{c|c} \psi_{\mathcal{F}} & 0 \\ \hline 0 & \hat{\phi} \end{array}\right) \sim_{\mathcal{F},\epsilon,stable\ \ uniq.} \left(\begin{array}{c|c} \psi_{\mathcal{F}} & 0 \\ \hline 0 & \hat{\psi} \end{array}\right) \approx_{\mathcal{F},\epsilon} \begin{pmatrix} & & \\ & \psi & \\ & & \end{pmatrix}?$$

But: The required size of $\mathcal{F}$ (corresponding to N) depends on $\hat{\phi}, \hat{\psi}, \psi_{\mathcal{F}}$, which depend on smallness of d, hence to the size of $\mathcal{F}$. Again the UCT sorts this.

Task:

- Make this work for "TA something".
- Does "TA something" cover all { separable, simple, unital, nuclear, UCT}$\otimes$ UHF?

Theorem. *The class { separable, simple, unital, nuclear, $\mathcal{Z}$-stable, UCT C^*-algebras } is classified by Ell($\cdot$).*

Problem: Do all separable, nuclear C^*-algebras satisfy the UCT? Now we quote some intermediate questions. If the above problem has affirmative answer then we may be able to answer them. Consider strongly self-absorbing / super-simple C^*-algebras.

Definition. *A unital, separable C^*-algebra $D \neq \mathbb{C}$ is called strongly self-absorbing (s.s.a) if there exists an $\phi : D \xrightarrow{\cong} D \otimes D$ such that $\phi \approx_{a.u} 1_D \otimes 1_D$.*

Example. *There is an hierarchy $\mathcal{Z}, M_{2^\infty}, M_{3^\infty}, \overset{un.\ many}{\ldots}, \mathcal{Q}, \mathcal{O}_\infty, \mathcal{O}_\infty \otimes M_{2^\infty}, \ldots, \mathcal{O}_\infty \otimes \mathcal{Q}, \mathcal{O}_2$.*

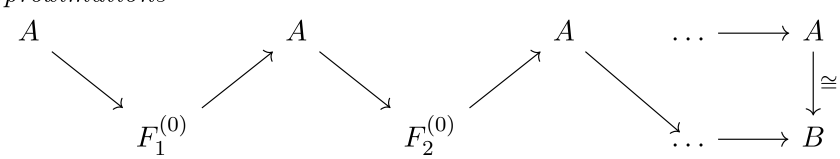
Questions:

- Are these all? Equivalently, does every s.s.a C^*-algebra satisfy the UCT?
- **Quasidiagonality question for s.s.a:** Is every s.s.a and finite C^*-algebra quasidiagonal? Equivalently, If D is s.s.a and finite, is it true that $D \hookrightarrow \mathcal{Q}$?
- **Infinite version:** If $D \neq \mathcal{O}_2$ is s.s.a, is it true that $D \hookrightarrow_{unital} \mathcal{Q} \otimes \mathcal{O}_\infty$? The answer is affirmative if D satisfies the UCT.
- Possible values for $K_0(D)$?
- Is there a Künneth formula for D? What is we assume that D is quasidiagonal or ∞-quasidiagonal?
- If D is quasidiagonal, s.s.a with $K_0(D) = \mathbb{Z}[\frac{1}{2}]$ and $K_1(D) = 0$, is it true that $D \cong M_{2^\infty}$?
- In particular assume that D is s.s.a such that $D \subset M_{2^\infty}$. Then if $K_0(D) = \mathbb{Z}[\frac{1}{2}]$ and $K_1(D) = 0$, is it true that either $D \cong M_{2^\infty}$ or $D \subset M_{p^\infty}$ (equivalently $D \cong \mathcal{Z}$)?

Remark: For a non-UCT example find a s.s.a D with torsion in $K_0(D)$ or $K_1(D) \neq 0$.

Question: How to recover (information about) a C^*-algebra from dim_{nuc} or decompositon rank (dr) approximately?

Example. *Consider $A = M_{2^\infty}$ or $A = \mathcal{Q}$. Then $drA = 0$. We get the c.p approximations*



where the arrows correspond to the c.p maps. Then A and B are isomorphic as operator systems. The idea is to recover information about A from B.

9. E-Theory and Extensions / Thomsen

E-Theory is the homotopy quotient of the theory of extensions, quote by Connes and Higson.

Definition. *Let A and B be C^*-algebras. An asymptotic morphism from A to B is a family $(\phi_t)_{t\in[0,\infty)}$ of maps from A to B with the following properties:*

- *the map $t \mapsto \phi_t(a)$ is continuous for every $a \in A$;*
- *the family (ϕ_t) is asymptotically multiplicative:*

$$\lim_{t\to\infty} \|\phi_t(ab) - \phi_t(a)\phi_t(b)\| = 0$$

for every $a, b \in A$, and the same holds for linearity and involution.

We say that two asymptotic morphisms (ϕ_t) and (ψ_t) are equivalent if

$$\lim_{t\to\infty} \|\phi_t(a) - \psi_t(a)\| = 0$$

for every $a \in A$.

A homotopy between asymptotic morphisms $(\phi_t^{(0)})$ and $(\phi_t^{(1)})$ is an asymptotic morphism (ϕ_t) from A to $C([0,1], B)$ such that $\phi_t(a)(0) = \phi_t^{(0)}$ and $\phi_t(a)(1) = \phi_t^{(1)}$ for all $t \in [0,\infty)$ and $a \in A$. We denote the set of homotopy classes of asymptotic morphisms from A to B by $[[A, B]]$.

An important feature of asymptotic morphisms is that they induce maps on K-theory. Let (ϕ_t) be an asymptotic morphism from A to B, and p a projection in A. Then for t sufficiently large, $\phi_t(p)$ is close to a projection q in B. The class of q in $Proj(B)$ is independent of the choices made, and eventually we get induced maps $\phi_* : K_0(A) \to K_0(B)$.

A very important example of asymptotic morphism is the Connes-Higson connecting morphism. Consider a short exact sequence of separable C^*-algebras,

$$\phi : 0 \to B \to E \xrightarrow{p} A \to 0$$

and choose a cross section s for p. Choose an approximate unit (u_t) for B which is quasicentral for E;

$$\lim_{t\to\infty} \|u_t e - e u_t\| = 0$$

for every $e \in E$. Then define $CH(\phi)_t : SA \to B$ by

$$CH(\phi)_t(f \otimes a) = f(u_t)s(a).$$

One can prove that this is indeed an asymptotic morphism since $(f(u_t))$ will still be quasicentral for E and hence

$$\lim_{t\to\infty} \|f(u_t)s(a)g(u_t)s(b) - (fg)(u_t)s(ab)\| =$$

$$\lim_{t\to\infty} \|f(u_t)s(a)g(u_t)s(b) - (fg)(u_t)s(a)s(b) + (fg)(u_t)(s(a)s(b) - s(ab))\| = 0$$

since $s(a)s(b) - s(ab) \in B$. The class $[CH] \in [[SA, B]]$ is independent of s (up to equivalence) and of (u_t).

If we consider asymptotic morphisms from A to $B \otimes \mathbb{K}$, we can define addition just like in Ext by fixing an isomorphism from $M_2(\mathbb{K})$ to $\mathbb{K}$. and taking the orthogonal sum. In this way $[[A, B \otimes \mathbb{K}]]$ is an abelian semigroup.

There is another addition if we consider asymptotic morphisms $(\phi_t), (\psi_t)$ from A to SB. We can move them homotopically to asymptotic morphisms (ϕ'_t) and (ψ'_t), whose ranges are supported on $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ respectively. Then $(\phi'_t + \psi'_t)$ is an asymptotic morphism from A to SB which depends only on the homotopy classes $[\phi]$ and $[\psi]$. This addition is not commutative in general but it coincides with the previous if B is stable.

An inverse for an asymptotic morphism (ϕ_t) from A to SB is given by $(\phi_t \circ (\rho \otimes id))$ where $\rho : S \to S$ is $\rho(f)(s) = f(1 - s)$.

Definition. *Let A and B be separable C^*-algebras. Then $E(A, B) := [[SA, SB \otimes \mathbb{K}]]$.*

Hence it natural to define a composition product $E(A, B) \times E(B, C) \to E(A, C)$. First we note that, any asymptotic morphism (ϕ_t) from A is equivalent to an asymptotic morphism (ϕ'_t) that is uniform on every norm-compact subset of A. Second, any two equivalent asymptotic morphisms are homotopic by the "straight line".

Theorem. *(1) If A, B and C are separable C^*-algebras and (ϕ_t) and (ψ_t) are uniform asymptotic morphisms from A to B and B to C respectively, then for any increasing parametrisation r of the interval $[0, \infty)$ that grows sufficiently quickly, the family $(\psi_{r(t)} \circ \phi_t)$ is an asymptotic morphism from A to C.*

(2) The resulting asymptotic morphism depends only on $[\phi]$ and $[\psi]$ and hence defines a composition $[[A, B]] \times [[B, C]] \to [[A, C]]$ that is associative and agrees with the ordinary composition for homomorphisms.

Thus there is an additive category $\mathbf{E}$ whose objects are C^*-algebras and the morphisms from A to B are the groups $E(A, B)$. One can prove that $E(\cdot, B)$ (resp. $E(A, \cdot)$) is a homotopy invariant, stable, half-exact covariant (resp. contravariant) and satisfies Bott periodicity. It is a universal functor in the sense that any functor on $\mathbf{SC}^*$ to an additive category $\mathbf{A}$ that is homotopy invariant, stable and half-exact factors uniquely through $\mathbf{E}$.

Finally, we note that if A is a separable, nuclear C^*-algebra, then $E(A, B)$ is naturally isomorphic with $KK(A, B)$ for every separable C^*-algebra B. The canonical functor from $KK(A, B)$ to $E(A, B)$ is given by the Connes-Higson morphism in the following way: $KK(A, B) \cong KK(SA, SB \otimes \mathbb{K}) \cong Ext(A, SB \otimes \mathbb{K}) \xrightarrow{CH} [[SA, SB \otimes \mathbb{K}]] = E(A, B)$. The inverse map is given by the universal property of $\mathbf{E}$. For every separable C^*-algebra D we have a pairing $KK(A, D) \times E(D, B) \to KK(A, B)$. For $D = A$, consider the homomorphism that we get from the pairing with 1_A and this will give an inverse.

Goal: To see if the Connes-Higson map $CH : Ext_h(A,B) \to [[SA,B]]$ is an isomorphism of semigroups for a general separable C^*-algebra A and a stable σ-unital C^*-algebra B. If it is not stable put $B \otimes \mathbb{K}$. The quote of Connes and Higson implies that CH is an isomorphism. Is it?

Note: As we have seen, if A is nuclear then $Ext(A,B)$ coincides with $Ext_h(A,B)$ and CH is an isomorphism. From now on A is separable and B is σ-unital and stable.

A quick recap on the notion of homotopy for extensions. Two extensions $0 \to B \to E_0 \to A \to 0$ and $0 \to B \to E_1 \to A \to 0$ are homotopic if there is a third one such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & {\scriptstyle ev_0}\uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & IB & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle ev_1} & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

In the Busby-invariant picture we have $\phi_0 : A \to \mathcal{Q}(B)$ and $\phi_1 : A \to \mathcal{Q}(B)$ respectively, and a homotopy between them is a Busby-invariant $\phi : A \to \mathcal{Q}(IB)$ such that evaluation at 0 and 1 gives ϕ_0 and ϕ_1.

Definition. *An extension $0 \to B \to E \xrightarrow{p} A \to 0$ is asymptotically split if there is an asymptotic morphism (ϕ_t) from A to E such that $p \circ \phi_t = id$ for all $t \in [0,\infty)$.*

We say that an extension is semi-invertible if it is stably asymptotically split.

Theorem. *When A is a suspension; meaning that $A = SD$ for some C^*-algebra D, all extensions of A by B are semi-invertible and hence $Ext_h(A,B)$ is a group.*

Theorem. *If A is a suspension, then CH is an isomorphism of groups.*

Define $Ext^{-\frac{1}{2}}(A,B) \hookrightarrow Ext_h(A,B)$ to be the group of semi-invertible extensions modulo the asymptotically split ones. So for $A = SD$, we have that $Ext^{-\frac{1}{2}}(A,B) = Ext_h(A,B)$ and $CH : Ext^{-\frac{1}{2}}(A,B) \to [[SA,B]]$ is an isomorphism of groups.

Question: Are all extensions semi-invertible?

Answer: No.

Question: Is CH an isomorphism?

Answer: No.

Example (Wassermann)**.** *There exists a representation $\pi : SL_3(\mathbb{Z}) \to B(H)$ such that the C^*-algebra $E = C^*(\pi(SL_3(\mathbb{Z})), \mathbb{K})$ defines a non semi-invertible extension*

$$\phi : 0 \to \mathbb{K} \to E \to E/\mathbb{K} \to 0.$$

Furthermore, this extension is non semi-invertible in homotopy; that is, in $Ext_h(A, B)$ *and also* $CH(\phi) = 0$. *As a result,* CH *is not an isomorphism.*

But on the other hand there is a big class of C^*-algebras A for which their extensions by B are all semi-invertible. For example, for every $n \in \mathbb{N}$ we get a group $Ext_h(C_r^*(\mathbb{F}_n), B)$ for stable B.

New question: Is it true that $Ext^{-\frac{1}{2}}(A, B) \cong Ext^{-1}(A, B)$ and $CH : Ext^{-\frac{1}{2}}(A, B) \to [[SA, B]]$ is an isomorphism?

The following is a partial answer since it gives a criterion to check if CH is an isomorphism.

Theorem (Manuilov, Thomsen)**.** *Let* A, A' *be* C^**-algebras such that there exist asymptotic morphisms* $\phi : A \to A, \lambda : A \to A', \mu : A' \to A$*, and that*

$$[id_A] + [\phi] = [\mu] \cdot [\lambda]$$

in $[[A, A]]$ *where* $\cdot$ *is the product of homotopy classes of asymptotic morphisms. Then if* $CH : Ext^{-\frac{1}{2}}(A', B) \to [[SA', B]]$ *is an isomorphism we have that* $CH : Ext^{-\frac{1}{2}}(A, B) \to [[SA, B]]$ *is also an isomorphism.*

Question: Is $CH : Ext_h(A, B) \to [[SA, B]]$ surjective?

Answer: The question is still open. But we can exactly describe the image. For that consider a quasicentral approximate unit $\{u_n\}_{n\in\mathbb{N}} \subset B$ for E such that $u_n u_{n-1} = u_{n-1}$ for all $n \in \mathbb{N}$. Let $s : A \to E$ be a cross-section for $p : E \to A$ and $f \in C_0(\mathbb{R}, A)$. Define

$$\phi_t(f) = \sum_{j=0}^{\infty} \Delta_j s(f(t - t_j))\Delta_j$$

where $\Delta_j = \sqrt{u_j - u_{j-1}}$, and $t_j \in [0, \infty)$ such that $\lim_{j\to\infty} t_j = \infty$ and $\lim_{j\to\infty}(t_j - t_{j-1}) = 0$.

Let $\tau_s(f)(t) = f(t - s)$ be the $\mathbb{R}$-action. Then

$$\phi_t(\tau_s(f)) = \phi_{t-s}(f) \;\; (*).$$

Denote the homotopy classes of asymptotic morphisms that satisfy $(*)$ by $[[SA, B]]_\tau$.

Theorem. *The Connes-Higson map* $CH : Ext_h(A, B) \to [[SA, B]]_\tau$ *is surjective.*

10. Trace Scaling automorphisms of $\mathcal{W} \otimes \mathbb{K}$ / Nawata

The algebra $\mathcal{W}$

$\mathcal{W}$ is a certain simple separable nuclear stably projectionless C^*-algebra with a unique tracial state τ and $K_0(\mathcal{W}) = K_1(\mathcal{W}) = 0$. $\mathcal{W}$ can be constructed as an inductive limit of type I C^*-algebras. It is also KK-equivalent to $\{0\}$ and $\mathcal{O}_2$.

Note: A C^*-algebra A is said to be stably projectionless if $A \otimes \mathbb{K}$ has no non-zero projections. In particular, every projectionless C^*-algebra is non-unital.

Properties of $\mathcal{W}$ (Razak)

(1) Every automorphism of $\mathcal{W}$ is approximately inner.
(2) $\mathcal{W} \cong \mathcal{W} \otimes M_{2^\infty}$
(Note that if A is a simple separable C^*-algebra satisfying the UCT with a unique tracial state and $dim_{nuc}(A) < \infty$, then $A \otimes \mathcal{W} \cong \mathcal{W}$ by the classification theorem of Gong-Lin and Elliott-Niu.)
(3) (Evans, Kishimoto-Kumjian, Dean, Robert)
There is a flow γ on $\mathcal{O}_2$ such that $\mathcal{W} \otimes \mathbb{K} \cong \mathcal{O}_2 \rtimes_\gamma \mathbb{R}$.
(4) If α is an automorphism of $\mathcal{W} \otimes \mathbb{K}$, then there is a positive real number $\lambda(\alpha)$ such that
$$\tau \otimes Tr \circ \alpha = \lambda(\alpha)\tau \otimes Tr$$
because $\mathcal{W}$ has a unique tracial state τ.
(5) For any $\lambda > 0$, there is an automorphism α on $\mathcal{W} \otimes \mathbb{K}$ such that $\lambda(\alpha) = \lambda$.
(6) Let $\alpha, \beta \in Aut(\mathcal{W} \otimes \mathbb{K})$. Then α is approximately unitarily equivalent to β if and only if $\lambda(\alpha) = \lambda(\beta)$.

Main results

Theorem. *Let α and β be automorphisms of $\mathcal{W} \otimes \mathbb{K}$ such that $\lambda(\alpha) \neq 1$ and $\lambda(\beta) \neq 1$. Then*
$$\alpha \sim_{o.c} \beta \Leftrightarrow \lambda(\alpha) = \lambda(\beta).$$

Note: We say that α is outer conjugate to β ($\alpha \sim_{o.c} \beta$) if there is $\gamma \in Aut(\mathcal{W} \otimes \mathbb{K})$ and $u \in U(M(\mathcal{W} \otimes \mathbb{K}))$ such that $\alpha = Ad(u) \circ \gamma \circ \beta \circ \gamma^{-1}$.

Theorem. *Let α and β be automorphisms of $\mathcal{W}$. Assume that α^n and β^n are strongly outer for any $n \in \mathbb{Z} \setminus \{0\}$. Then α is outer conjugate to β.*

Strategy

(I) Rokhlin type theorem (Projections in $A^\omega \cap A'$)
(II) Homotopy type theorem (Unitaries in $A^\omega \cap A'$)

$$\text{(I)+(II)} \xRightarrow{Herman-Ocneanu} \text{the stability of } \alpha \xRightarrow{EK-intertwining\ \ arg.} \text{Main Thm.}$$

If A is stably projectionsless, then $A^\omega \cap A'$ is stably projectionless.

Kirchberg's central sequence C^*-algebras

Let ω : free ultrafilter on $\mathbb{N}$ and put

$$A^\omega := \ell^\infty(\mathbb{N}, A)/\{(a_n)_n : \lim_{n\to\omega} \|a_n\| = 0\}$$

and $A_\omega := A^\omega \cap A'$. Also, let

$$Ann(A, A^\omega) := \{(a_n)_n \in A_\omega : \lim_{n\to\omega} \|a_n b\| = 0 \text{ for any } b \in A\}.$$

Then $Ann(A, A^\omega)$ is an ideal of A_ω and define

$$F(A) := A_\omega / Ann(A, A^\omega).$$

If A is unital, then $F(A) = A_\omega$. Now, let $\{h_n\}_{n\in\mathbb{N}}$ be an approximate unit for A. Then $[(h_n)_n] = 1$ in $F(A)$. We have that $F(A \otimes \mathbb{K}) \cong F(A)$.

Every automorphism α of A induces an automorphism of $F(A)$. We denote it by the same symbol α.

Rokhlin type theorem

Theorem. *Let α be an automorphism of $\mathcal{W} \otimes \mathbb{K}$ such that $\lambda(\alpha) \neq 1$. Then α has the Rokhlin property; that is, for any $k \in \mathbb{N}$, there exist projections $\{e_{1,0}, e_{1,1}, \dots, e1, k-1, e_{2,0}, \dots, e_{2,k}\}$ in $F(\mathcal{W} \otimes \mathbb{K})$ such that*

- $\sum_{j=0}^{k-1} e_{1,j} + \sum_{j=0}^{k} e_{2,j} = 1$;
- $\alpha(e_{1,j}) = e_{1,j+1}$;
- $\alpha(e_{2,j}) = e_{2,j+1}$.

Theorem. *Let α be an automorphism of $\mathcal{W}$. Assume that α^n is strongly outer for any $n \in \mathbb{Z} \setminus \{0\}$. Then α has the Rokhlin property.*

Homotopy type theorem

Theorem. *Let u be a unitary element in $F(\mathcal{W})$. Then there exists a continuous path of unitaries $U : [0,1] \to F(\mathcal{W})$ such that*

- $U(0) = 1$;
- $U(1) = u$;
- $Lip(U) \leq 2\pi$.

Lemma. *Let u and v be unitaries in $F(\mathcal{W})$ such that $\tau_\omega(f(u)) > 0$ and $\tau_\omega(f(v)) > 0$ for any $f \in C(\mathbb{T})_+ \setminus \{0\}$. Then there exists some unitary $w \in F(\mathcal{W})$ such that*

$$wuw^* = v \Leftrightarrow \tau_\omega(f(u)) = \tau_\omega(f(v))$$

for every $f \in C((T)$.

Properties of $F(\mathcal{W})$

(I) (Essentially shown by Matui and Sato)
$F(\mathcal{W})$ has a unique tracial state τ_ω. It also has strict comparison; that is, for any $a, b \in F(\mathcal{W})_+$,

$$d_{\tau_\omega}(a) < d_{\tau_\omega}(b) \Rightarrow \exists r \in F(\mathcal{W}) \text{ s.t } rbr^* = a.$$

(II) Let $a \in F(\mathcal{W})_+$ such that $d_{\tau_\omega}(a) > 0$. For any $t \in [0, d_{\tau_\omega}(a))$, there is a projection $p \in \overline{aF(\mathcal{W})a}$ such that $\tau_\omega(p) = t$.

(III) Let p and q be projections in $F(\mathcal{W})$. Assume that $\tau_\omega(p) < 1$ and $\tau_\omega(1) < 1$. Then

$$p \sim_{MvN} q \Leftrightarrow p \sim_u q.$$

11. Partitions of unity and the Toms-Winter Conjecture/ Castillejos

Toms-Winter Conjecture: Let A be a simple, separable, unital, infinite dimensional, nuclear C^*-algebra. The following are equivalent:

(1) $dim_{nuc} A < \infty$
(2) $A \otimes \mathcal{Z} \cong A$
(3) A has strict comparison.

There is a W^* analog of this conjecture. First we give some definitions. Let M be a von Neumann algebra with separable predual.

Definition. *M is injective if for some faithful representation $\pi : M \to B(H)$ there exists a conditional expectation $E : B(H) \to M$.*

Remark: Nuclearity in this analogy is the C^*-version of injectivity.

Definition. *M is hyperfinite if there exists an increasing family of finite dimensional von Neumann algebras $\{F_n\}_{\in \mathbb{N}}$ such that $M = \overline{\cup_n F_n}^{SOT}$.*

Remark: The C^*-version of it is finite nuclear dimension.

Definition. *$M \subset B(H)$ is a factor if the center $Z(M) = M' \cap M = \mathbb{C}1_M$.*

Remark: The C^*-version of it is simplicity.

Definition. *A factor M is of type II_1 if there exists a faithful normal (ultra weakly continuous) trace.*

Example. $R = \overline{\odot_{n=1}^{\infty} M_2}^{SOT}$ *is the only hyperfinite II_1 factor.*

Theorem (Connes)**.** *If M is an injective II_1 factor then M is hyperfinite; $M \cong R$.*

The idea of the proof is:

(1) M is hyperfinite
(2) M is McDuff; $M \overline{\otimes} R \cong M$
(3) M is injective

and (3) $\Rightarrow$ (2) $\Rightarrow$ (1). We see that this is the analog of the Toms-Winter conjecture.

Key Step: Assume a unique trace T. Then we can produce a map $M \to R^\omega = \ell^\infty(R)/\{\{x_n\} : \lim_{n\to\omega} T(x_n^* x_n) = 0\}$.

Note: For the case of C^*-algebras, assume that $T \in \partial_e T(A)$ (extreme trace). Then $\pi_T(A)'' \cong R$ and we can produce a map $A \to A_\omega / J_{A_\omega} \cong R^\omega$, where $J_{A_\omega} = \{(a_n) \in \ell^\infty(A) : \lim_{n\to\omega} T(a_n^* a_n) = 0\}$. When $T(A)$ is a Bauer simplex; that is, $T(A) \neq \varnothing$ and $\partial_e T(A)$ is compact, we can do the above procedure in a continuous way by forming the bundle $C(\partial_e T(A), R)$.

Definition. *A II_1 factor M has property Γ if $M^\omega \cap M' \neq \mathbb{C}1$.*

Proposition (Connes)**.** *If M is injective then it has property Γ.*

Remark: R has property Γ , while $L\mathbb{F}_2$ does not. Therefore, $R \ncong L\mathbb{F}_2$.

Theorem (Dixmier)**.** *A II_1 factor M has property Γ if and only if for every $n \in \mathbb{N}$ there are pairwise orthogonal projections $p_1, \ldots, p_n \in M^\omega \cap M'$ such that $T_{M^\omega}(p_i) = \frac{1}{n}$.*

Observation:

- $T(p_i x) = \frac{1}{n} T(x)$ for every $x \in M$.
- If M has property Γ, then $M_n(\mathbb{C}) \hookrightarrow M^\omega$ in such a way that $D_n(\mathbb{C}) \subset M^\omega \cap M'$.

Remark: McDuff factors have property Γ. More generally, if M has property Γ, then $M \overline{\otimes} N$ has property Γ for any N. However, property Γ does not imply McDuff property.

Definition. *Let A be a separable, unital C^*-algebra with $T(A) \neq \varnothing$. We say that A has property Γ if for each $n \in \mathbb{N}$ there exist pairwise orthogonal positive $e_1, \ldots, e_n \in A_\omega \cap A'$ such that $T(ae_i) = \frac{1}{n} T(a)$ for every $a \in A$ and $T \in T(A_\omega)$.*

Example. *$\mathcal{Q}, \mathcal{Z}, \mathcal{Z}$-stable C^*-algebras have property Γ.*

A new partition of unity argument

Definition. *Let A be a separable, unital C^*-algebra with $T(A) \neq \varnothing$. We say that A has complemented tracial orthogonal partition of unity ($CPoU$) if for every family of positive $a_1, \ldots, a_k \in A$ and $\delta > 0$ such that*

$$sup_{T \in T(A)} min_{i=1,\ldots,k} T(a_i) < \delta$$

(for every $T \in T(A)$ there are a_i such that $T(a_i) < \delta$), there exist pairwise orthogonal positive contractions $e_1, \ldots, e_k \in A_\omega \cap A'$ satisfying

$$\pi(\sum e_i) = 1 \ \ and \ \ \tau(a_i e_i) < \delta T(e_i).$$

Theorem. *Let A be a simple, separable, unital, nuclear C^*-algebra with property Γ. Then A has a CPoU.*

Coming back to Toms-Winter conjecture we consider the following corollary.

Corollary. *A simple, separable, unital, nuclear and $\mathcal{Z}$-stable C^*-algebra has a CPoU.*

Theorem. *A simple, separable, unital, nuclear and $\mathcal{Z}$-stable C^*-algebra A has $dim_{nuc} A \leq 1$.*

Lemma. *Let A be separable, unital, nuclear C^*-algebra with $T(A) \neq \varnothing$ and a CPoU with no finite dimensional representation. Then for every $n \in \mathbb{N}$ there exists a unital embedding $M_n(\mathbb{C}) \hookrightarrow A_\omega \cap A' / J_{A_\omega}$.*

Corollary. *Such a C^*-algebra if it has the property Γ and strict comparison, then it is $\mathcal{Z}$-stable.*

Theorem. *Let A be a simple, separable, unital, infinite dimensional, nuclear C^*-algebra. The following are equivalent.*

(1) $dim_{nuc} A < \infty$
(2) $A \otimes \mathcal{Z} \cong A$
(3) A has strict comparison and property Γ.

12. UCT and groupoid C^*-algebras / Renault

Let G be locally compact, Hausdorff groupoid with a Haar system. Jean-Louis Tu in 1999 proved that if G is topologically amenable, or more generally, if it satisfies Haagerup's property, then its C*-algebra $C^*(G)$ satisfies the UCT. We will present the main ideas of the proof.

Theorem (Tu, 99)**.** *Let G be a σ-compact locally compact Hausdorff groupoid with Haar system. If there exists a proper G-affine euclidean bundle, then its C^*-algebras (full or reduced) satisfy the UCT.*

The existence of a proper G-affine euclidean bundle is the Haagerup property. This property is satisfied in particular when G is topologically amenable.

Strategy of the proof

The proof of the UCT, as well as the Baum-Connes conjecture hinges on the construction of a G-algebra A and elements $\alpha \in KK_G(C_0(G^{(0)}), A)$ and $\beta \in KK_G(A, C_0(G^{(0)}))$ such that

$$\alpha \otimes_A \beta = 1 \quad \text{in} \quad KK_G(C_0(G^{(0)}), C_0(G^{(0)}))$$

Applying the descent functor, one obtains that $C^*(G)$ is KK-subequivalent to the crossed product C*-algebra $A \rtimes G$. Since $A \rtimes G$ has the UCT, one concludes through the easy

Lemma. *The UCT is preserved under KK-subequivalence.*

G-spaces and G-bundles

A space Z is G- proper if $G \star Z \to Z \times Z$, $(\gamma, z) \mapsto (\gamma z, z)$ is proper; that is, for every $L, M \subset Z$ compact, the set $\{\gamma \in G : \ \gamma L \cap M \neq \varnothing\}$ is compact. Then the quotient space Z/G is locally compact Hausdorff in the quotient topology and the quotient map is open.

We define the groupoid G to be proper if $G^{(0)}$ is a proper G-space. Hence $G \ltimes Z$ is proper if Z is a proper G-space.

Let X be a locally compact topological space. A $C_0(X)$-algebra is a pair (A, θ) consisting of a C^*-algebra A and a homomorphism $\theta : C_0(X) \to ZM(A)$ such that $\theta(C_0(X))A = A$. Such an algebra can be canonically identified with the algebra of continuous sections of a bundle having for fibre over $x \in X$, the algebra $A_x = A/C_xA$ where $C_xA = \{f \in C_0(X) : f(x) = 0\}$.

Let A be a $C_0(G^{(0)})$-algebra. The action of G on A is given by C^*-algebra isomorphisms $a_\gamma : A_{s(\gamma)} \to A_{r(\gamma)}$, for every $\gamma \in G$, such that for any composable pair $(\gamma, \gamma') \in G^2$ we have $a_{\gamma\gamma'} = a_\gamma a_{\gamma'}$. We call such an algebra a G-algebra. Note that a $G \ltimes Z$-algebra is a G-algebra.

Definition. *A G-algebra A is proper if it is a $G \ltimes Z$-algebra for some proper G-space Z.*

Euclidean Affine Space

An Euclidean space is seen as an affine space E on which the vector space $\vec{E}$ acts by translations: $E \times \vec{E} \to E$ by $(e + \vec{a}) \mapsto \vec{a}$. Consider a family of euclidean affine spaces $E = \coprod_{x \in X} E_x$ over a locally compact Hausdorff topological space X in a way that we get a continuous bundle $p : E \to X$. In this way $p : \vec{E} \to X$ is a Hilbert bundle.

Suppose that $X = G^{(0)}$ and consider an action of G on E and a unitary representation L on $\vec{E}$ such that $E_{s(\gamma)} \to E_{r(\gamma)}$, $\gamma(e + \vec{a}) = \gamma e + L(\gamma)\vec{a}$. We say that a subset M of E is bounded if there exists a continuous section a of $p : E \to G^{(0)}$ and $R > 0$ such that for all $e \in M$, $d(e, a(p(e)) \leq R$.

Definition. *E is called G-proper if for all $L, M \subset E$ bounded, the set $\{\gamma \in G : \ \gamma L \cap M \neq \varnothing\}$ is relatively compact.*

Choose a section $x \to e_x \in E_x$ of p. Then $\gamma e_{s(\gamma)} = e_{r(\gamma)} + c(\gamma)$ where $c \in Z^n(G, p^*E)$ (cocycle) such that $c(\gamma\gamma') = c(\gamma) + L(\gamma)c(\gamma')$.

Definition. *The groupoid G has satisfies the Haagerup's property if there exists a proper Euclidean affine G-bundle.*

U. Haagerup has shown that the free groups have this property by showing the existence of a conditionally negative type function which is proper.

Amenable groupoids

Let (G, λ) be be a locally compact groupoid with Haar system. Its regular G-Hilbert bundle $L^2(G, \lambda)$ is defined by its fibers $L^2(G^x, \lambda^x)$, its fundamental family of continuous sections $C_c(G)$ and the action $L(\gamma)f(\gamma_1) = f(\gamma^{-1}\gamma_1)$.

Definition. *One says that G is amenable if there exists a sequence (ξ_n) in $C_c(G)^+$ which, viewed as sections of the Hilbert bundle $L^2(G, \lambda)$, satisfy*

(i) $\|\xi_n(x)\| \to 1$ *uniformly on compact subsets of* $G^{(0)}$*;*
(ii) $\|L(\gamma)\xi_n(s(\gamma)) - \xi_n(r(\gamma))\| \to 0$ *uniformly on compact subsets of* G.

Amenability implies Haagerup Property

One considers the regular real Hilbert bundle with infinite multiplicity $\vec{E} = L^2(G, \lambda) \otimes \ell^2(\mathbb{N})$ with fibers $L^2(G^x, \lambda^x) \otimes \ell^2(\mathbb{N})$ and linear action $L \otimes 1$.

One chooses

(i) an increasing sequence (K_n) of compact subsets exhausting $G^{(0)}$;
(ii) an increasing sequence (L_n) of compact subsets exhausting G;
(iii) a sequence (ξ_n) in $C_c(G)^+$ such that $\|\xi_n(x)\| = 1$ for $x \in K_n$ and $\|L(\gamma)\xi_n(s(\gamma)) - \xi_n(r(\gamma))\| \leq 1/n$ for $\gamma \in L_n$.

Then,

$$c(\gamma) = \sum_{\mathbb{N}} (L(\gamma)\xi_{n,s(\gamma)} - \xi_{n,r(\gamma)}) \otimes e_n$$

belongs to $\vec{E}_{r(\gamma)}$ and $c : G \to p^*\vec{E}$ is a one-cocycle, i.e. it satisfies

$$c(\gamma\gamma') = c(\gamma) + (L(\gamma) \otimes 1)c(\gamma')$$

One checks easily that c is proper: for $r > 0$, the set of γ's such that $\|c(\gamma)\| \leq r$ is compact. Then,

$$A(\gamma)\xi = (L(\gamma) \otimes 1)\xi + c(\gamma)$$

defines a proper affine isometric action of G on $\vec{E}$.

The space Z of an euclidean affine space

Let E be an euclidean affine space. If E is not finite dimensional, it is not locally compact and $C_0(E)$ is not suitable. However, it has a substitute which is defined as an inductive limit. The main observation is that for an inclusion $U \subset V$ of finite dimensional subspaces, the map

$$\begin{aligned} \pi_{U,V} \ : \ & \mathbb{R}_+ \times V \to \mathbb{R}_+ \times U \\ & (t, v) \mapsto (\sqrt{t^2 + \|\vec{uv}\|^2}, u) \end{aligned}$$

where u is the orthogonal projection of v on U, is proper and these maps define a projective system.

Definition. *The space Z of E is the projective limit of this projective system.*

As a set, it can be identified to $\mathbb{R}_+ \times E$ through similar maps $\pi_{U,E}$.

A proper G-space

This construction is still valid when E be an euclidean affine bundle over a locally compact space X and produces a locally compact space $Z = \mathbb{R}_+ \times E$. If E is a G-euclidean affine bundle, we define $\gamma(e,t) = (\gamma e, t)$ for $\gamma \in G$ and $(e,t) \in Z$.

Lemma. *Let E is a G-euclidean affine bundle, then*

(i) Z is a G-space;

(ii) if E is a proper G-euclidean affine bundle, then Z is a proper G-space.

The algebra of an euclidean affine space

Again, when the euclidean affine space E is not finite dimensional, algebras such that $C_0(E, \mathrm{Cliff}(E))$ cannot be defined. Here is a substitute defined as an inductive limit. To each finite dimensional subspace U, one associates the graded C*-algebra

$$\mathcal{C}(U) = C_0(T^*U, \mathrm{End}(\Lambda^*\vec{U}) \otimes \mathbb{C})),$$

where $\mathrm{End}(\Lambda^*\vec{U})$ is graded by the even and odd degree and its graded suspension

$$\mathcal{SC}(U) = \mathcal{S}\hat{\otimes}\mathcal{C}(U)$$

where $\mathcal{S} = C_0(\mathbb{R})$ is graded by the even and odd functions.

The inclusion

An inclusion $U \subset V$ of finite dimensional subspaces gives an injective $*$-homomorphism $j_{V,U} : \mathcal{C}(U) \to \mathcal{A}(V)$ as follows.
The orthogonal decomposition $V = U + \vec{W}$ gives the tensor decomposition $\mathcal{A}(V) = \mathcal{A}(U)\hat{\otimes}\mathcal{A}(\vec{W})$.
The inclusion $j_{V,U}$ is essentially given by the $*$-homomorphism

$$\begin{aligned} \varphi_W \ : \ C_0(\mathbb{R}) &\to C_0(\mathbb{R})\hat{\otimes}\mathcal{A}(W) \\ f &\mapsto f(X\hat{\otimes}1 + 1\hat{\otimes}c) \end{aligned}$$

where X and c are unbounded multipliers: X is the function $t \mapsto t$ on $\mathbb{R}$ and, for $w \in T^*W = W \otimes \mathbb{C}$, $c(w) = \mathrm{ext}(w) + \mathrm{ext}(w)^*$ is the complex Clifford multiplication.

The inductive limit

One can check that the system is inductive. This provides the C*-algebra $A = A(E) = \varinjlim_U \mathcal{SA}(U)$ of the euclidean affine space E.
Since $C_0(\mathbb{R}_+ \times T^*U) = C_0(\mathbb{R})_{\mathrm{even}} \otimes C_0(T^*U, \mathbb{C})$ embeds into

$$\mathcal{SA}(U) = C_0(\mathbb{R})\hat{\otimes}C_0(T^*U, \mathrm{End}(\Lambda^*\vec{U}) \otimes \mathbb{C}))$$

as a central subalgebra and the embedding $j_{V,U}$ is compatible with π_{T^*U,T^*V}, A is a Z-algebra, where Z is the locally compact space constructed from the euclidean affine space T^*E as above. The same construction applies when E is a G-euclidean affine bundle and provides a $G \ltimes Z$-algebra A. Therefore, if G acts properly on E, then A is a proper G-algebra.

$A \rtimes G$ **belongs to** $\mathcal{N}$

Lemma. *Let A be a G-algebra, where the fibers of A are type I and G is a proper groupoid with Haar system. Then the crossed product $A \rtimes G$ is a type I C^*-algebra.*

Proposition. *Let $A = A(E)$ be the G-algebra constructed from a proper euclidean affine G-bundle E. Then the crossed product $A \rtimes G$ is an inductive limit of type I C^*-algebras.*

Idea of the proof: functional calculus

More details: We may assume that $G^{(0)} = Z$ and that G is proper. From a G-invariant strictly positive compact operator T, one constructs by functional calculus an increasing sequence of G-invariant finite-dimensional subspaces (V_n) which exhaust E and defines $A_n = \mathcal{S}\hat{\otimes}\mathcal{C}(V_n)$. It is a type I

C*-algebra and it union is dense in A. Then $A_n \rtimes G$ is type I and its union is dense in $A \rtimes G$.

the KK-theory elements η and D

The unbounded multiplier X of $\mathcal{S}$ defines an unbounded multiplier of the A-Hilbert module A which has a compact resolvent: this defines $\eta \in KK_G(C_0(G^{(0)}), A)$. The element $D \in KK_G(A, C_0(G^{(0)}))$ is constructed from a continuous field of G-algebras (A_t) over $[0, \infty]$ where $A_\infty = A$ and A_t is an algebra of compact operators for t finite and an unbounded multiplier (D_t) with $D_0 = 0$ and D_∞ is the Clifford multiplication c.

13. Villadsen algebras and some regularity conditions / Bosa

Definition. *Let A be a C^*-algebra. We may say that $a, b \in A_+$ are Cuntz subequivalent ($a \preceq b$) if there exists $\{x_n\}_{n\in\mathbb{N}} \subset A$ such that*

$$\|x_n b x_n^* - a\| \to 0.$$

We say that a, b are Cuntz equivalent ($a \sim b$) if $a \preceq b$ and $b \preceq a$. We define

$$Cu(A) = (A \otimes \mathbb{K})_+ / \sim$$

and with $[a] + [b] = [a \otimes b]$ it becomes an abelian monoid with partial order $[a] \leq [b] \Leftrightarrow a \preceq b$.

Theorem. *Cu is continuous; $Cu(\varinjlim A_i) = \varinjlim Cu(A_i)$.*

Note: We consider the limit in a new category where the Cuntz semigroup lies.

Definition. *Given a separable, nuclear, simple C^*-algebra A, the Cuntz semigroup $Cu(A)$ has w-comparison if given $x, y_1, \ldots, y_n \ldots$ such that $x <_s y_j$ (there is $k_j \in \mathbb{N} : (k_j + 1)x < k_j y)$) for every j, we have $x \leq \sum_{j=1}^{\infty} y_j$.*

Definition. *Given a separable, nuclear, simple C^*-algebra A, the Cuntz semigroup $Cu(A)$ has the Corona Factorisation Property (CFP) if given $x, y_1, \ldots, y_n \ldots$ there exists $m \in \mathbb{N}$ such that if $x \leq m y_j$ for all j, we have $x \leq \sum_{j=1}^{\infty} y_j$.*

Theorem. *Under our assumptions w-comparison implies CFP.*

Example. *The other direction is not true in general. Consider $[0, 1] \cup \{\infty\}$. Here we have $x + y > 1 \Rightarrow x + y = \infty$. This satisfies CFP. However, if we choose $x = \frac{3}{4}$ and $y_i = \frac{1}{2^{i+2}}$ we get $x <_s y_j$ for all j but $x \not\leq \sum_{j=1}^{\infty} y_j$.*

Villadsen algebras of type I and II

Type I: unital, simple AH-algebras with $sr A = 1$. (Inspired by Rørdam's and Tom's examples)

Type II: unital, simple AH-algebras with unique trace and can get any stable rank and real rank. In particular, they are not $\mathcal{Z}$-stable.

Definition. *Let X, Y be compact Hausdorff spaces. A $*$-homomorphism $\phi : C(X) \otimes \mathbb{K} \to C(Y) \otimes \mathbb{K}$ is called diagonal if there exist $k \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_k : Y \to X$ and natural orthogonal projections $p_1, \ldots, p_k \in C(Y) \otimes \mathbb{K}$ such that $\phi = (id_{C(Y)} \otimes a) \circ (\hat{\phi} \otimes id_{\mathbb{K}})$ where $a : \mathbb{K} \otimes \mathbb{K} \xrightarrow{\cong} \mathbb{K}$ and $\hat{\phi}(f) = \sum_{i=1}^{k} (f \circ \lambda_i) p_i$.*

Denote by $\mathfrak{k} : \mathbb{N}_\infty \times \mathbb{N} \to \mathbb{N}$,

$$(k, n) \mapsto \begin{cases} kn!n, & k < \infty \\ n^2 n!, & k = \infty. \end{cases}$$

- **k$\neq\infty$:** Denote $X_n^k := \mathbb{D}^k \times \mathbb{C}P^{\mathfrak{k}(k,1)} \times \ldots \times \mathbb{C}P^{\mathfrak{k}(k,n)}$.
- **k=∞:** Denote $X_n^\infty := \mathbb{D}^{n^2 n!} \times \mathbb{C}P^{\mathfrak{k}(\infty,1)} \times \ldots \times \mathbb{C}P^{\mathfrak{k}(\infty,n)}$.
- We have that $X_0^k = \mathbb{D}^k$, hence $X_n^k = X_{n-1}^k \times \mathbb{C}P^{\mathfrak{k}(k,n)}$.
- We have that $X_0^\infty = \mathbb{D}$, hence $X_n^\infty = \mathbb{D}^{n^2 n! - (n-1)^3 (n-1)^2} \times X_{n-1}^\infty \times \mathbb{C}P^{\mathfrak{k}(\infty,n)}$.

Denote the canonical projections by $\pi_{k,n}^1 : X_n^k \to X_{n-1}^k$ and $\pi_{k,n}^2 : X_n^k \to \mathbb{C}P^{\mathfrak{k}(k,n)}$. Define the diagonal maps $\widetilde{\phi}_n^k : C(X_n^k) \otimes \mathbb{K} \to C(X_{n+1}^k) \otimes \mathbb{K}$ arising from the tuple $(\pi_{k,n+1}^1, \theta_1) \cup (y_{n,j}^{(k)}, \zeta_{n+1}^{(k)})_{j=1}^{n+1}$ where $(y_{n,j}^{(k)})_{j=1}^{n+1}$ are chosen so that the resulting C^*-algebra is simple and θ_1 is the trivial line bundle. Let $p_0^k \in C_0(X^k) \otimes \mathbb{K}$ be a projection of rank 1 and $p_n^k = \widetilde{\phi}_{n,0}(p_0^k)$. Define $A_n^k = p_n^k (C(X_n^k) \otimes \mathbb{K}) p_n^k$ and $\phi_n^k = \widetilde{\phi}_n^k|_{A_n^k}$. Then the Villadsen algebra is given by the inductive limit

$$\mathcal{V}_k = \varinjlim (A_n^k, \phi_n^k).$$

Consider any vector bundle η over X_i^k and one has

$$(\phi_i^k)^*(\eta) \cong \pi_{k,i+1}^{1*}(\eta) \oplus (i+1) rank(\eta) \zeta_{i+1}^k,$$

where $(\phi_i^k)^*$ denotes the induced map from isomoprhism classes of vector bundles over X_i^k to isomorphism classes of vector bundles over X_{i+1}^k.

Theorem. *(Villadsen) For each $k \in \mathbb{N}_\infty$*

- *$\mathcal{V}_k$ has unique trace;*
- *$sr(\mathcal{V}_k) = k+1$ and $sr(\mathcal{V}_\infty = \infty$;*
- *$k < RR(\mathcal{V}_k) < k+1$.*

Theorem. *(Bosa - Christensen) For the class of Villadsen algebras of type II, w-comparison is equivalent with the CFP. Also,*

- *$\mathcal{V}_k$ has w-comparison for every $k \neq \infty$.*
- *$\mathcal{V}_\infty$ does not have CFP.*

Lemma. *Let $k \in \mathbb{N}_\infty$. For each $n \in \mathbb{N}$ there exist projections $e_n, q_1^{(n)}, \ldots, q_n^{(n)} \in \mathcal{V}_k \otimes \mathbb{K}$ such that*

(1) $e_n \preceq q_i^{(n)} \oplus q_i^{(n)}$ for all $i = 1, \ldots, n$;
(2) $e_n \not\preceq q_1^{(n)} \oplus \ldots \oplus q_n^{(n)}$;
(3) $\tau(q_1^{(n)} \oplus \ldots \oplus q_n^{(n)}) \to_n k$ and $\tau(e_n) \to_n 0$.

References

[BS17] Selçuk Barlak and Gábor Szabó. "Approaching the UCT problem via crossed products of the Razak-Jacelon algebra". In: (2017). URL: arxiv.org/abs/1712.00823.

[BL17a] Selçuk Barlak and Xin Li. "Cartan subalgebras and the UCT problem". In: *Adv. Math.* 316 (2017), pp. 748–769. ISSN: 0001-8708. URL: https://doi.org/10.1016/j.aim.2017.06.024.

[BL17b] Selçuk Barlak and Xin Li. "Cartan subalgebras and the UCT problem, II". In: (2017). URL: arxiv.org/abs/1704.04939.

[EG16] Caleb Eckhardt and Elizabeth Gillaspy. "Irreducible representations of nilpotent groups generate classifiable C^*-algebras". In: *Münster J. Math.* 9.1 (2016), pp. 253–261. ISSN: 1867-5778.

[EGM18] Caleb Eckhardt, Elizabeth Gillaspy, and Paul McKenney. "Finite decomposition rank for virtually nilpotent groups". In: (2018). URL: arXiv:1706.04142,%202017.

[EKM15] Caleb Eckhardt, Craig Kleski, and Paul McKenney. "Classification of C*-algebras generated by representations of the unitriangular group $UT(4, \mathbb{Z})$". In: 271 (June 2015).

[EM14] Caleb Eckhardt and Paul McKenney. "Finitely generated nilpotent group C*-algebras have finite nuclear dimension". In: 2015 (Sept. 2014).

[CM84] A. L Carey and W. Moran. "Characters of nilpotent groups". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 96.1 (1984), pp. 123–137. DOI: 10.1017/S0305004100062009.

[Chr16] Martin S. Christensen. *Regularity of Villadsen algebras and characters on their central sequence algebras*. arXiv:1603.01166v4. 2016. eprint: 1603.01166.

[Dad94] Marius Dadarlat. "Shape theory and asymptotic morphisms for C^*-algebras". In: *Duke Math. J.* 73.3 (1994), pp. 687–711.

[DL96] Marius Dadarlat and Terry A. Loring. "A universal multicoefficient theorem for the Kasparov groups". In: *Duke Math. J.* 84.2 (1996), pp. 355–377. ISSN: 0012-7094. URL: https://doi.org/10.1215/S0012-7094-96-08412-4.

[Gab17] James Gabe. "A new proof of Kirchberg's $\mathcal{O}_2$-stable classification". In: (2017). URL: arxiv.org/abs/1706.03690.

[HK01] Nigel Higson and Gennadi Kasparov. "E-theory and KK-theory for groups which act properly and isometrically on Hilbert space". In: 144 (Apr. 2001), pp. 23–74.

[HO17] Hyun Ho Lee and Hiroyuki Osaka. "Tracially sequentially-split *-homomorphisms between C^*-algebras". In: (2017). URL: arxiv.org/abs/1707.07377.

[Lan73] Christopher Lance. "On nuclear C-algebras". In: *Journal of Functional Analysis* 12.2 (1973), pp. 157–176. ISSN: 0022-1236. DOI: https://doi.org/10.1016/0022-1236(73)90021-9. URL:

http://www.sciencedirect.com/science/article/pii/0022123673900219.

[MT04] Vladimir Manuilov and Klaus Thomsen. "The Connes-Higson construction is an isomorphism". In: *J. Funct. Anal.* 213.1 (2004), pp. 154–175.

[MT07] V. Manuilov and K. Thomsen. "On the lack of inverses to C^*-extensions related to property groups". In: *Canad. Math. Bull.* 50.2 (2007), pp. 268–283.

[MT11a] Vladimir Manuilov and Klaus Thomsen. "Semi-invertible extensions of C^*-algebras". In: *J. Funct. Anal.* 260.5 (2011), pp. 1446–1462.

[MT11b] Vladimir Manuilov and Klaus Thomsen. "Shape theory and extensions of C^*-algebras". In: *J. Lond. Math. Soc. (2)* 84.1 (2011), pp. 183–203.

[MN10] Ralf Meyer and Ryszard Nest. "Homological algebra in bivariant K-theory and other triangulated categories. I". In: *Triangulated categories*. Vol. 375. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2010, pp. 236–289.

[MR76] Calvin C. Moore and Jonathan Rosenberg. "Groups with T1 primitive ideal spaces". In: *Journal of Functional Analysis* 22.3 (1976), pp. 204–224. ISSN: 0022-1236. DOI: https://doi.org/10.1016/0022-1236(76)90009-4. URL: www.sciencedirect.com/science/article/pii/0022123676900094.

[Naw17] Norio Nawata. *Trace scaling automorphisms of* $\mathcal{W} \otimes \mathbb{K}$. 2017. eprint: 1704.02414.

[RS86] Jonathan Rosenberg and Claude Schochet. "The Künneth theorem and the universal coefficient theorem for equivariant K-theory and KK-theory". In: *Mem. Amer. Math. Soc.* 62.348 (1986), pp. vi+95. ISSN: 0065-9266. URL: https://doi.org/10.1090/memo/0348.

[RS87] Jonathan Rosenberg and Claude Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor*, Duke Math. J. **55** (1987), 431–474, DOI:10.1215/S0012-7094-87-05524-4.

[Sch82] Claude Schochet. "Topological methods for C^*-algebras. II. Geometric resolutions and the Künneth formula". In: *Pacific J. Math.* 98.2 (1982), pp. 443–458. ISSN: 0030-8730. URL: http://projecteuclid.org/euclid.pjm/1102734267.

[TWW17] Aaron Tikuisis, Stuart White, and Wilhelm Winter, *Quasidiagonality of nuclear* C*-*algebras*, Ann. Math. (2) **185** (2017), no. 1, 229–284, DOI:10.4007/annals.2017.185.1.4. ↑

[Tu99] Jean-Louis Tu. "La conjecture de Baum-Connes pour les feuilletages moyennables". In: *K-Theory* 17.3 (1999), pp. 215–264. ISSN: 0920-3036. URL: doi.org/10.1023/A:1007744304422.

[Win16] Wilhelm Winter. "QDQ vs. UCT". In: *In Operator Algebras and Applications:* 12 (2016), pp. 321–342.

[Win17] Wilhelm Winter. *Structure of nuclear C*-algebras: From quasidiagonality to classification, and back again.* arXiv:1712.00247v1. 2017. eprint: 1712.00247.

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